FOR THE
IB DIPLOMA

Mathematics

ANALYSIS AND APPROACHES HL

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WORKED SOLUTIONS

DYNAMIC LEARNING

HODDER EDUCATION

1 Counting principles

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 1A

13 Choose cat: ${}^5C_1 = 5$
and
Choose dog: ${}^{11}C_1 = 11$
Total choices: $5 \times 11 = 55$

14 Choose man: ${}^7C_1 = 7$
and
Choose woman: ${}^4C_1 = 4$
Total choices: $7 \times 4 = 28$

15 **a** Choose Y9: ${}^{95}C_1 = 95$
and
Choose Y10: ${}^{92}C_1 = 92$
and…
Total choices: $95 \times 92 \times 86 \times 115 \times 121 = 1.05 \times 10^{10}$

b Choose Y9 or Y10: ${}^{95}C_1 + {}^{92}C_1 = 187$
and
Choose Y11: ${}^{86}C_1 = 86$
and
Choose Y12: ${}^{115}C_1 = 115$
and
Choose Y13: ${}^{121}C_1 = 121$
Total choices: $187 \times 86 \times 115 \times 121 = 223\,781\,030$

16 **a** Choose S: ${}^5C_1 = 5$
and
Choose M: ${}^8C_1 = 8$
and
Choose D: ${}^6C_1 = 6$
Total choices: $5 \times 8 \times 6 = 240$

b Choose M: ${}^8C_1 = 8$
and
Choose S or D: ${}^5C_1 + {}^6C_1 = 11$
Total choices: $8 \times 11 = 88$

c Choose S and choose M: ${}^5C_1 \times {}^8C_1 = 40$
or
Choose S and choose D: ${}^5C_1 \times {}^6C_1 = 30$
or
Choose M and choose D: ${}^8C_1 \times {}^6C_1 = 48$
Total choices: $40 + 30 + 48 = 118$

17 Arrange 6 boys: $6! = 720$
and
Arrange 5 girls: $5! = 120$
Total arrangements: $720 \times 120 = 86\,400$

18 **a** Arrange 7 distinct digits: $7! = 5040$ arrangements
b Those divisible by 5 must end in 5;
Arrange 6 distinct digits for the left end of the number:
$6! = 720$ arrrangements

19 **a** Arrange 8 plants: $8! = 40\,320$
b Arrange 2 tulips: $2! = 2$
and
Arrange 6 roses: $6! = 720$
Total arrangements: $2 \times 720 = 1\,440$

20 Choose 6 from 10: ${}^{10}C_6 = 210$

21 **a** Choose 3 from 7: ${}^{7}C_3 = 35$
b Choose 2 from 6: ${}^{6}C_2 = 15$

22 Arrange 4 from 9: ${}^{9}P_4 = 3\,024$

23 Arrange 3 from 8: ${}^{8}P_3 = 336$

24 Arrange 2 from 17: ${}^{17}P_2 = 272$

25 **a** Choose shirt: ${}^{9}C_1 = 9$
and
Choose trousers: ${}^{6}C_1 = 6$
and
Choose waistcoat: ${}^{4}C_1 = 4$
Total choices: $9 \times 6 \times 4 = 216$
b Number of ways to have blue and green:
Choose green shirt: ${}^{3}C_1 = 3$
and
(Choose blue trousers and any waistcoat: ${}^{2}C_1 \times {}^{4}C_1 = 8$
or
Choose non-blue trousers and blue waistcoat: ${}^{4}C_1 \times 1 = 4$)
Total choices: $3 \times (8 + 4) = 36$
Total choices excluding green and blue: $216 - 36 = 180$

26 a Arrange 5 digits: $5! = 120$
b Select first digit from 1,2,3: 3
and
Arrange remaining 4 digits: $4! = 24$
Total arrangements: $3 \times 24 = 72$

27 Choose 5 numbers from 40: ${}^{40}C_5 = 658\,008$
and
Choose 2 numbers from 10: ${}^{10}C_2 = 45$
Total choices: $658\,008 \times 45 = 29\,610\,360$

28 Choose a goalkeeper: ${}^{3}C_1 = 3$
and
Choose 3 defenders from 6: ${}^{6}C_3 = 20$
and
Choose 5 midfielders from 8: ${}^{8}C_5 = 56$
and
Choose 2 forwards from 4: ${}^{4}C_2 = 6$
Total choices: $3 \times 20 \times 56 \times 6 = 20\,160$

29 Choose 15 from 48 to go to Paris: ${}^{48}C_{15}$
and
Choose 12 from the remaining 33 to go to Rome: ${}^{33}C_{12}$
and
Choose 10 from the remaining 21 to go to Athens: ${}^{21}C_{10}$
Total choices: ${}^{48}C_{15} \times {}^{33}C_{12} \times {}^{21}C_{10} = 1.37 \times 10^{26}$

30 **a** Choose 2 boys from 17: ${}^{17}C_2 = 136$
and
Choose 3 girls from 15: ${}^{15}C_3 = 455$
Total choices: $136 \times 455 = 61\,880$

b Choose 2 boys from 17: ${}^{17}C_2 = 136$
and
Choose 1 from Baha and Connie: ${}^{2}C_1 = 2$
and
Choose 2 girls from 13: ${}^{13}C_2 = 78$
Total choices: $136 \times 2 \times 78 = 21\,216$

31 Choose Gold or Silver for Usain: ${}^{2}C_1 = 2$
and
For the remaining two places
Arrange 2 from 7 athletes: ${}^{7}P_2 = 42$
Total arrangements: $2 \times 42 = 84$

32 Arrange 3 girls from 16: ${}^{16}P_3 = 3\,360$
and
Arrange 2 boys from 14: ${}^{14}P_2 = 182$
Total arrangements: $3\,360 \times 182 = 611\,520$

33 Three letter/four digit:
Arrange 3 letters from 26 and arrange 4 digits from 9:
${}^{26}P_3 \times {}^{9}P_4 = 47\,174\,400$
Four letter/three digit:
Arrange 4 letters from 26 and arrange 3 digits from 9:
${}^{26}P_4 \times {}^{9}P_3 = 180\,835\,200$
Total codes: $47\,174\,400 + 180\,835\,200 = 228\,009\,600$

34 **a** Choose 5 students from 15: ${}^{15}C_5 = 3\,003$

bi Choose 4 students from 14: ${}^{14}C_4 = 1\,001$

bii All-boy committee: Choose 5 students from 7: ${}^{7}C_5 = 21$
All other committees must contain at least one girl: $3\,003 - 21 = 2\,982$

c $\frac{2\,982}{3\,003} = 0.993$

35 ${}^{n}C_2 = \frac{n(n-1)}{2} = 210$
$n^2 - n = 420$
$n^2 - n - 420 = 0$
$(n - 21)(n + 20) = 0$
$n = 21$ (reject the solution $n = -20$)

36 ${}^{n}P_2 = n(n-1) = 132$
$n^2 - n - 132 = 0$
$(n - 12)(n + 11) = 0$
$n = 12$ (reject the solution $n = -11$)

37 $^nP_n = n!$

$^nP_{n-1} = \frac{n!}{1!} = n!$

So $^nP_n = {}^nP_{n-1}$

^nP_n is the number of ways to arrange n different items

$^nP_{n-1}$ is the number of ways to arrange $n-1$ of n different items; but in doing so, the unused item is also specified, so the 'unused' is effectively the final unit of the ordering. We can reason that the number of possible arrangements of $n-1$ of n items must be the same as the number of complete arrangements.

38 Assuming that neither the presents nor the boxes are considered identical:

a Total ways: $2^5 = 32$

b Total ways: $3^4 = 81$

39 Arrange 6 students from 18 to be the front row: $^{18}P_6$

and

Arrange 6 students from 12 to be the middle row: $^{12}P_6$

and

Arrange 6 students from 6 to be the back row: $^6P_6 = 6!$

Total arrangements: $^{18}P_6 \times {}^{12}P_6 \times {}^6P_6 = 6.40 \times 10^{15}$

40 **a** Each triangle must have three points as vertices.

Each set of three vertices can make exactly one triangle (since no three points are collinear).

The total number of triangles that can be formed is $^{10}C_3 = 120$

b Each set of four vertices can make exactly one convex quadrilateral (if points $ABCD$ form a quadrilateral then $ACBD$ would not be considered a quadrilateral).

The total number of quadrilaterals that can be formed is $^{10}C_4 = 210$

41 Each handshake requires a pair of people (order not important)

Number of handshakes is $^nC_2 = 465$

$\frac{n(n-1)}{2} = 465$

$n^2 - n = 930$

$n^2 - n - 930 = 0$

$(n-31)(n+30) = 0$

$n = 31$ (reject the solution $n = -30$)

Exercise 1B

1 Arrange 9 units and internally arrange one unit consisting of B and C: $9! \times 2! = 725\,760$

2 Arrange 7 units and internally arrange one unit consisting of 3 SL books: $7! \times 3! = 30\,240$

3 Freely arrange 6 letters: $6! = 720$

Fix A as the start letter and arrange 5 letters: $5! = 120$

Total not starting with A: $720 - 120 = 600$

Tip: Alternatively, in a fully symmetrical problem like this, you could argue that of the 720 free arrangements, $\frac{5}{6}$ of them will not start with A, and obtain the answer $\frac{5}{6} \times 720 = 600$

4 Freely arrange 7 digits: $7! = 5\,040$

Fix 67 as the end digits and arrange 5 digits: $5! = 120$

Total not ending with 67: $5\,040 - 120 = 4\,920$

5 Freely choose 4 from 12 toys: ${}^{12}C_4 = 495$
Choose 4 from 7 soft toys: ${}^{7}C_4 = 35$
Total choices which are not all soft toys: $395 - 35 = 460$

6 Freely arrange 7 letters: $7! = 5\,040$
Arrange COM and then arrange PUTE: $3! \times 4! = 144$
Total arrangements not beginning with letters C,O,M in some order: $5\,040 - 144 = 4\,896$

7 a Choose dark and milk and white: ${}^{10}C_1 \times {}^{8}C_1 \times {}^{7}C_1 = 560$
b Freely choose 3 from 25 chocolates: ${}^{25}C_3 = 2\,300$
Choose 3 from 10 dark chocolates: ${}^{10}C_3 = 120$
Total choices not all dark: $2\,300 - 120 = 2\,180$

8 Free arrangements: $7! = 5\,040$
D and then arrange 6 letters: $6! = 720$
Arrange 6 letters and then A: $6! = 720$
D and then arrange 5 letters and then A: $5! = 120$
Total arrangements: $5\,040 - 720 - 720 + 120 = 3\,720$

9 Freely arrange 3 letters from 7: ${}^{7}P_3 = 210$
Arrange 3 letters from the 4 consonants: ${}^{4}P_3 = 24$
Arrangements which are not all consonants: $210 - 24 = 186$

10 Free choice of 8 students from 29: ${}^{29}C_8 = 4\,292\,145$
Choice of 8 students from 13 boys: ${}^{13}C_8 = 1\,287$
Choice of 1 girl from 16 and 7 boys from 13:
${}^{16}C_1 \times {}^{13}C_7 = 16 \times 1\,716 = 27\,456$
Total choice of 8 students with at least 2 girls:
$4\,292\,145 - 1\,287 - 27\,456 = 4\,263\,402$

11 Arrange 3 types of chocolate: 3!
and
Arrange 7 types of milk chocolate: 7!
and
Arrange 5 types of white chocolate: 5!
and
Arrange 4 types of dark chocolate: 4!
Total arrangements: $3! \times 7! \times 5! \times 4! = 87\,091\,200$

12 Arrange 10 students from 12A and 12B: 10!
and
Choose 3 of the 11 gaps: ${}^{11}C_3$
and
Arrange the 3 students from 12C into those 3 gaps: 3!
Total permitted arrangements: $10! \times {}^{11}C_3 \times 3! = 3\,592\,512\,000$

13 Freely arrange 9 digits: 9!
Arrange the 4 even digits: 4!
and
Arrange the 5 odd digits into the 5 gaps: 5!
Total arrangements with no two odd numbers next to each other: $4! \times 5!$
Probability that no two odd numbers are next to each other in a random arrangement:
$\frac{4!\times 5!}{9!} = \frac{1}{126}$

14 **a** Arrange the two parents: $2!$
and
Arrange the four children between them: $4!$
Total arrangements: $2! \times 4! = 48$

b Arrange the two parents: $2!$
and
Arrange the five 'units': $5!$
Total arrangements: $2! \times 5! = 240$

c Freely arrange the 6 people: $6! = 720$
Probability of **b** not occurring in a free arrangement:
$\frac{720-240}{720} = \frac{2}{3}$

15 Free choice of 5 cards from 52: ${}^{52}C_5$

a Choice of 5 cards from 13 spades: ${}^{13}C_5$
Probability that all are spades in a random deal: $\frac{{}^{13}C_5}{{}^{52}C_5} = \frac{33}{66\,640}$

b Choice of 5 cards from 26 red cards: ${}^{26}C_5$
Probability that all are red in a random deal: $\frac{{}^{26}C_5}{{}^{52}C_5} = \frac{253}{9\,996}$

c Choice of 1 black card from 26 and 4 red cards from 26: ${}^{26}C_1 \times {}^{26}C_4$
Probability that exactly one is black in a random deal: $\frac{{}^{26}C_1 \times {}^{26}C_4}{{}^{52}C_5} = \frac{1\,495}{9\,996}$
So probability of at least two black cards: $1 - \frac{253}{9\,996} - \frac{1\,495}{9\,996} = \frac{2\,062}{2\,499}$

16 Choose which seat they will take as the left-most of the 6 from a choice of 15:
${}^{15}C_1 = 15$
Arrange the 6 people: $6! = 720$
$15 \times 720 = 10\,800$

17 Free choice of 6 players from 15: ${}^{15}C_6 = 5\,005$
Not permitting:
Choice of 6 players from Team A only: ${}^{8}C_6 = 28$
Choice of 6 players from Team B only: ${}^{7}C_6 = 7$
Choice of 1 player from Team A and 5 players from Team B: ${}^{8}C_1 \times {}^{7}C_5 = 168$
Total permitted choices: $5\,005 - (28 + 7 + 168) = 4\,802$

18 Free choice of 7 tiles from 26: ${}^{26}C_7$
Not permitting:
Choice of 7 letters from 21 consonants: ${}^{21}C_7$
Choice of 1 letter from 5 vowels and 6 letters from 21 consonants: ${}^{5}C_1 \times {}^{21}C_6$
Total choices with at least 2 vowels: ${}^{26}C_7 - ({}^{21}C_7 + {}^{5}C_1 \times {}^{21}C_6) = 270\,200$

Mixed Practice

1 Arrange 3 athletes out of 8: ${}^{8}P_3 = 336$

2 Arrange 4 letters out of 26 and 3 digits out of 9: ${}^{26}P_4 \times {}^{9}P_3 = 180\,835\,200$

3 Choose 3 out of 15 girls and 3 out of 11 boys: ${}^{15}C_3 \times {}^{11}C_3 = 75\,075$

4 Freely arrange 5 cards: $5! = 120$
Only one of these arrangements would match the order of the envelopes.
Probability that all cards and envelopes match is $\frac{1}{120}$

5 Arrange 3 teachers and then arrange 8 students: $3! \times 8! = 241\,920$

6 **a** Choose 6 from 19 students: ${}^{19}C_6 = 27\,132$
b Choose Jack and Jill, then choose 4 from 17 students: ${}^{17}C_4 = 2380$
Probability of Jack and Jill both being chosen is $\frac{2380}{27132} = \frac{5}{57}$

7 Each of the three digits can take any of 9 values: $9^3 = 729$

8 Choose a goalkeeper: ${}^{3}C_1 = 3$
and
Choose 4 defenders from 7: ${}^{7}C_4 = 35$
and
Choose 4 midfielders from 8: ${}^{8}C_4 = 70$
and
Choose 2 forwards from 4: ${}^{4}C_2 = 6$
Total choices: $3 \times 35 \times 70 \times 6 = 44\,100$

9 Choose 3 toys from 7 for the youngest: ${}^{7}C_3 = 35$
and
Choose 2 toys from 4 for the eldest: ${}^{4}C_2 = 6$
and
Choose 2 toys from 2 for the middle child: ${}^{2}C_2 = 1$
Total choices: $35 \times 6 \times 1 = 210$

10 Choose 1 consonant from 5 to be the final letter: ${}^{5}C_1 = 5$
Choose 1 consonant from 4 to be the first letter: ${}^{4}C_1 = 4$
Freely arrange the middle 5 letters: $5! = 120$
Total arrangements: $5 \times 4 \times 120 = 2400$

11 Freely choose 5 students from 28: ${}^{28}C_5 = 98\,280$
Not permitting:
Select the two youngest and then choose 3 students from 26: ${}^{26}C_3 = 2\,600$
Total permitted choices: $98\,280 - 2\,600 = 95\,680$

12 Free choice of 7 students from 19: ${}^{19}C_7$
Not permitted:
Choice of 7 students from 9 boys: ${}^{9}C_7$
Choice of 7 students from 10 girls: ${}^{10}C_7$
Total permitted choices: ${}^{19}C_7 - {}^{9}C_7 - {}^{10}C_7 = 50\,232$

13 **a** Free choice of 6 students from 15: ${}^{15}C_6 = 5\,005$
bi Free choice of 6 students from 14: ${}^{14}C_6 = 3\,003$
bii Not permitted:
Choice of 6 students from 9 boys: ${}^{9}C_6 = 84$
Choice of 1 students from 6 girls and 5 from 9 boys: ${}^{6}C_1 \times {}^{9}C_5 = 756$
Total permitted choices: $5\,005 - 84 - 756 = 4\,165$
c $\frac{4\,165}{5\,005} = 0.832$

14 Choose which driver is assigned to which vehicle: ${}^{2}C_1 = 2$
and
Choose 3 from 10 to be passengers in the car: ${}^{10}C_3 = 120$
and
Choose 7 from 7 to be passengers in the SUV: ${}^{7}C_7 = 1$
Total choices: $2 \times 120 \times 1 = 240$

15 **a** Free arrangement of 7 letters: $7! = 5040$

b Arrange 3 vowels to be a single unit: $3! = 6$
and
Arrange 5 units D, M, S, T, AEO: $5! = 120$
Total arrangements: $6 \times 120 = 720$

c Arrange 4 consonants: $4! = 24$
and
Arrange 3 vowels: $3! = 6$
and
Choose 3 of the 5 spaces (including ends) among the consonants:
${}^{5}C_3 = 10$
Total arrangements: $24 \times 6 \times 10 = 1\,440$

16 **a** Choose which end Amit will stand: ${}^{2}C_1 = 2$
and
Arrange the other four people: $4! = 24$
Total arrangements with Amit at one end: $2 \times 24 = 48$

b Freely arrange 5 people: $5! = 120$
Total arrangements with Amit not at an end: $120 - 48 = 72$

c Arrangements with Amit at the left: $1 \times 4! = 24$
Arrangements with Ed at the right: $4! \times 1 = 24$
Arrangements with both conditions: $1 \times 3! \times 1 = 6$.
These cases will have been counted in each of the first two arrangements, so the double-count must be corrected.
The total number with either or both conditions is $24 + 24 - 6 = 42$

17 **a** Probability of correctly matching all 6 numbers is $\frac{1}{{}^{50}C_6} = \frac{1}{15\,890\,700}$

b Total number of choices is ${}^{50}C_6 = 15\,890\,700$
Number of choices matching 0 of the lottery numbers: ${}^{44}C_6$
Number of choices matching 1 of the lottery numbers: ${}^{6}C_1 \times {}^{44}C_5$
Number of choices matching 2 of the lottery numbers: ${}^{6}C_2 \times {}^{44}C_4$
Then the number of ways to get a prize is
${}^{50}C_6 - ({}^{44}C_6 + {}^{6}C_1 \times {}^{44}C_5 + {}^{6}C_2 \times {}^{44}C_4)$
Probability of winning a prize:
$\frac{{}^{50}C_6 - ({}^{44}C_6 + {}^{6}C_1 \times {}^{44}C_5 + {}^{6}C_2 \times {}^{44}C_4)}{{}^{50}C_6} = \frac{279\,335}{15\,890\,700} = \frac{347}{19\,740}$

18 ${}^{n}P_2 = n(n-1) = 380$
$n^2 - n - 380 = 0$
$(n - 20)(n + 19) = 0$
$n = 20$ (rejecting the negative root)

19 **a** Arrange 3 girls as a single unit: $3!$
and
Arrange 4 units (3 boys and the block of girls): $4!$
Total permitted arrangements: $3! \times 4! = 144$

b Arrange 3 boys: 3!
and
Choose 3 of the four spaces (including the ends) for the girls: ${}^{4}C_3 = 4$
and
Arrange 3 girls into the chosen spaces: 3!
Total permitted arrangements: $3! \times 4 \times 3! = 144$

20 **a** If the largest is 5, 6 or 7, then this is equivalent to drawing 4 cards from $\{1, 2, 3, 4, 5, 6, 7\}$ and excluding the single case of drawing 1, 2, 3 or 4.
Number of possible choices: ${}^{7}C_4 - 1 = 34$

b Free choice of 4 cards: ${}^{9}C_4 = 126$
Not permitted:
Choice of all odd numbers: ${}^{5}C_4 = 5$
Choice of one even number and three odd numbers: ${}^{4}C_1 \times {}^{5}C_3 = 40$
Total number of permitted choices: $126 - (5 + 40) = 81$

21 Choose one driver from 3: ${}^{3}C_1 = 3$
and
(designating the empty seat as a passenger) arrange the 7 other seats to the seven 'passengers': 7!
Total possible seating arrangements: $3 \times 7! = 15\,120$

22 **a** Arrange 6 people among 10 seats: ${}^{10}P_6 = 151\,200$

b Either the person with a cold must sit at an end, next to an empty seat:
Choose one end: ${}^{2}C_1 = 2$
Then arrange the other 5 people into the remaining 8 available spaces: ${}^{8}P_5 = 6\,720$
Total arrangements: $2 \times 6\,720 = 13\,440$
Or the person with a cold sits with an empty seat to either side:
Person with a cold, with an empty seat either side, represents a single unit
Arrange the 6 'units' into the 8 available slots: ${}^{8}P_6 = 20\,160$
Total possible arrangements: $13\,440 + 20\,160 = 33\,600$

23 **a** Choose 6 from 12 to be team A: ${}^{12}C_6 = 924$
This will count each split twice (since choosing 6 to be team A is equivalent to choosing the other 6 to be team A; the team letters are irrelevant, only the split matters).
So the total possible ways to divide into two teams of 6 is
$924 \div 2 = 462$

b Choose 4 from 12 to be team A: ${}^{12}C_4 = 495$
and
Choose 4 from the remaining 8 to be team B: ${}^{8}C_4 = 70$
The remainder will be team C.
Total ways to assign to teams A, B and C: $495 \times 70 = 34\,650$
However, since again the team labels are not relevant, this will count each split 3! times (since each split could be assigned A/B/C in 3! Ways)
So the total possible ways to divide into three teams of 4 is
$34\,650 \div 3! = 5775$

2 Algebra

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 2A

9

$$(1-2x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-2x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-2x)^3 + \cdots$$
$$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots$$

10

$$\left(1-\frac{1}{4}x\right)^{-3} = 1 + (-3)\left(-\frac{1}{4}x\right) + \frac{(-3)(-4)}{2!}\left(-\frac{1}{4}x\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(-\frac{1}{4}x\right)^3 + \cdots$$
$$= 1 + \frac{3}{4}x + \frac{3}{8}x^2 + \frac{5}{32}x^3 + \cdots$$

11

$$4(8+x)^{-\frac{1}{3}} = 2\left(1+\frac{1}{8}x\right)^{-\frac{1}{3}}$$
$$= 2\left(1 + \left(-\frac{1}{3}\right)\left(\frac{1}{8}x\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\frac{1}{8}x\right)^2 + \cdots\right)$$
$$= 2\left(1 - \frac{1}{24}x + \frac{1}{288}x^2 + \cdots\right)$$
$$= 2 - \frac{1}{12}x + \frac{1}{144}x^2 + \cdots$$

12

$$(2-5x)^{-1} = \frac{1}{2}\left(1-\frac{5}{2}x\right)^{-1}$$
$$= \frac{1}{2}\left(1 + (-1)\left(-\frac{5}{2}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{5}{2}x\right)^2 + \cdots\right)$$
$$= \frac{1}{2} + \frac{5}{4}x + \frac{25}{8}x^2 + \cdots$$

13 **a**

$$(9+x)^{\frac{1}{2}} = 3\left(1+\frac{1}{9}x\right)^{\frac{1}{2}}$$
$$= 3\left(1+\left(\frac{1}{2}\right)\left(\frac{1}{9}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{1}{9}x\right)^2+\cdots\right)$$
$$= 3+\frac{1}{6}x-\frac{1}{216}x^2 \ldots$$

b Expansion is valid for $\left|\frac{1}{9}x\right| < 1$

That is $|x| < 9$

c When $x = 1$ (which is within the interval of validity) $(9+x)^{\frac{1}{2}} = \sqrt{10}$:

$$(9+1)^{\frac{1}{2}} \approx 3+\frac{1}{6}(1)-\frac{1}{216}(1)^2 \ldots$$
$$\approx 3\frac{35}{216}$$
$$\approx 3.16204$$

14 **a**

$$(8-3x)^{\frac{1}{3}} = 2\left(1-\frac{3}{8}x\right)^{\frac{1}{3}}$$
$$= 2\left(1+\left(\frac{1}{3}\right)\left(-\frac{3}{8}x\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{3}{8}x\right)^2+\cdots\right)$$
$$= 2-\frac{1}{4}x-\frac{1}{32}x^2+\cdots$$

b Expansion is valid for $\left|-\frac{3}{8}x\right| < 1$

That is $|x| < \frac{8}{3}$

c When $x = 1$ (which is within the interval of validity), $\sqrt[3]{8-3x} = \sqrt[3]{5}$:

$$(8-3(1))^{\frac{1}{3}} \approx 2-\frac{1}{4}(1)-\frac{1}{32}(1)^2+\cdots.$$
$$\approx \frac{55}{32}$$
$$\approx 1.71875$$

15 **a**

$$x(1+3x)^{-2} = x\left(1+(-2)(3x)+\frac{(-2)(-3)}{2!}(3x)^2+\cdots\right)$$
$$= x-6x^2+27x^3+\cdots$$

b Expansion is valid for $|3x| < 1$

That is $|x| < \frac{1}{3}$

16

$$(1+x)(1-x)^{-1} = (1+x)\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\cdots\right)$$
$$= (1+x)(1+x+x^2+\cdots)$$
$$= 1+2x+2x^2+\cdots$$

17

$$(2+x)(1+6x)^{\frac{1}{4}} = (2+x)\left(1+\left(\frac{1}{4}\right)(6x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(6x)^2+\cdots\right)$$
$$= (2+x)\left(1+\frac{3}{2}x-\frac{27}{8}x^2+\cdots\right)$$
$$= 2+4x-\frac{21}{4}x^2+\cdots$$

The coefficient of x^2 is $-\frac{21}{4}$

18

$$(1+x)^2(1+2x)^{-2} = (1+2x+x^2)\left(1+(-2)(2x)+\frac{(-2)(-3)}{2!}(2x)^2+\cdots\right)$$
$$= (1+2x+x^2)(1-4x+12x^2+\cdots)$$
$$= 1-2x+5x^2+\cdots$$

The coefficient of x^2 is 5

19 **a**

$$(1-4x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(-4x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-4x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-4x)^3+\cdots$$
$$= 1-2x-2x^2-4x^3+\cdots$$

b Expansion is valid for $|-4x| < 1$

That is $|x| < \frac{1}{4}$

c When $x = 0.02, \sqrt{1-4x} = \sqrt{0.92} = \sqrt{23 \times \frac{4}{100}} = 0.2\sqrt{23}$

From the expansion in part **a**:

$$\sqrt{1-4\times 0.02} \approx 1-2(0.02)-2(0.02)^2-4(0.02)^3$$
$$\approx 1-0.04-0.0008-0.000032$$
$$\approx 0.959168$$
$$\sqrt{23} \approx 5\times 0.959168 = 4.79584$$

20

$$(1+ax)^{-3} = 1+(-3)(ax)+\frac{(-3)(-4)}{2!}(ax)^2+\frac{(-3)(-4)(-5)}{3!}(ax)^3+\cdots$$
$$= 1-3ax+6a^2x^2-10a^3x^3+\cdots$$

$-640 = -10a^3$

$a^3 = 64$

$a = 4$

21 **a**

Tip: You will have to do multiple expansions for this problem. The straightforward way is to take the product of the expansions for $(1+2x)^{\frac{1}{3}}$ and $(1-x)^{-\frac{1}{3}}$. In the worked solution below, you can see an alternative using a nested expansion. Either method will generate the same expansion, though the first way is more straightforward when considering the interval of validity.

$$\begin{aligned}\left(\frac{1+2x}{1-x}\right)^{\frac{1}{3}} &= \left(1+\frac{3x}{1-x}\right)^{\frac{1}{3}}\\ &= \left(1+3x(1-x)^{-1}\right)^{\frac{1}{3}}\\ &= \left(1+3x\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\cdots\right)\right)^{\frac{1}{3}}\\ &= (1+3x+3x^2+3x^3+\cdots)^{\frac{1}{3}}\\ &= 1+\left(\frac{1}{3}\right)(3x+3x^2+3x^3)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(3x+3x^2+3x^3)^2\\ &\quad+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(3x+3x^2+3x^3)^3\ldots\\ &= 1+x+x^2+x^3-(x^2+2x^3+\cdots)+\left(\frac{5}{3}x^3+\cdots\right)+\cdots\\ &= 1+x+\frac{2}{3}x^3+\cdots\end{aligned}$$

b When $x=0.04$, $\sqrt[3]{\frac{1+2x}{1-x}}=\sqrt[3]{\frac{1.08}{0.96}}=\sqrt[3]{\frac{0.12\times 9}{0.12\times 8}}=\frac{1}{2}\sqrt[3]{9}$

From the expansion in part **a**:

$$\begin{aligned}\frac{1}{2}\sqrt[3]{9} &\approx 1+0.04+\frac{2}{3}(0.04)^3\\ &\approx 1+0.04+.0000427\\ &\approx 1.0400427\end{aligned}$$

So $\sqrt[3]{9}\approx 2.08009$

22

$$\begin{aligned}(1+ax)^n &= 1+nax+\frac{n(n-1)}{2!}(ax)^2+\frac{n(n-1)(n-2)}{3!}(ax)^3+\cdots\\ &= 1-4x+9x^2+bx^3+\cdots\end{aligned}$$

Comparing coefficients:

$x^1: na=-4$ (1)

$x^2: \frac{n^2-n}{2}a^2=9$ (2)

$x^3: \frac{n(n-1)(n-2)}{3!}a^3=b$ (3)

$(1)^2: n^2a^2=16$ (4)

$2(2): n^2a^2-na^2=18$ (5)

$(4)-(5): na^2=-2$ (6)

$(4)/(6): n=-8$

So $a = \frac{1}{2}$

Then (3): $b = \frac{(-8)(-9)(-10)}{6}\left(\frac{1}{2}\right)^3 = -15$

23

$(1-x)(1+ax)^n$

$$= (1-x)\left(1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \cdots\right)$$
$$= 1 + x(na-1) + x^2\left(\frac{n(n-1)}{2}a^2 - na\right)$$
$$+ x^3\left(\frac{n(n-1)(n-2)}{6}a^3 - \frac{n(n-1)}{2}a^2 + \cdots\right) + \cdots$$
$$= 1 + x^2 + bx^3 + \cdots$$

Comparing coefficients:

$x^1: na - 1 = 0$ (1)

$x^2: \frac{n^2-n}{2}a^2 - na = 1$ (2)

$x^3: \frac{n(n-1)(n-2)}{3!}a^3 - \frac{n(n-1)}{2}a^2 = b$ (3)

(1): $na = 1$ (4)

2(2): $n^2a^2 - na^2 - 2na = 2$ (5)

$(5) + 2(4) - (4)^2: -na^2 = 3$ (6)

(6)/(4): $-a = 3$

So $a = -3, n = -\frac{1}{3}$

Then (3): $b = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{6}(-3)^3 - \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}(-3)^2$

$= \frac{14}{3} - 2$

$= \frac{8}{3}$

24

$$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \cdots$$
$$= 1 + 21x + bx^2 + bx^3 + \cdots$$

Comparing coefficients:

$x^1: na = 21$ (1)

$x^2: \frac{n(n-1)}{2}a^2 = b$ (2)

$x^3: \frac{n(n-1)(n-2)}{3!}a^3 = b$ (3)

Equating (2) and (3):

$6b = n(n-1)(n-2)a^3 = 3n(n-1)a^2$

$n(n-1)a^2[(n-2)a - 3] = 0$

From (1) reject $n = 0$

If $n = 1$ then $a = 21$ and $b = 0$, also rejected

Otherwise, $(n-2)a - 3 = 0$

Substituting (1):
$2a = na - 3 = 18$
So $a = 9, n = \frac{7}{3}$

Then $b = \frac{n(n-1)}{2}a^2 = \frac{\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)}{2}9^2 = 126$

Exercise 2B

General tip for exercise: Throughout this exercise there are two main methods which could be used:

Worked examples 2.4 and 2.5 use substitution: values for x are substituted in turn to eliminate each of the constants of the partial equation form, generating simultaneous equations. Ideally, the values chosen for x eliminate one of the constants altogether, making the working much easier.

An alternative approach is to compare coefficients. While this is not shown in the chapter examples, there are situations where this is preferable, for example, in a more complicated problem where the denominator has many factors, comparing coefficients allows for a check that the factoring was correct and complete.

Both methods are shown for question 5, and the method of comparing coefficients is again used in the worked solution to question 14.

5

$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$ for some constants A and B
Multiplying through by the denominator on the left:
$5x + 1 = A(x+2) + B(x-1)$

Method 1: Using substitution:
$x = 1: 6 = 3A$ so $A = 2$
$x = -2: -9 = -3B$ so $B = 3$

Method 2: Comparing coefficients:
$x^0: 1 = 2A - B$ (1)
$x^1: 5 = A + B$ (2)

$(1) + (2): 3A = 6$ so $A = 2$
$(2): B = 5 - A = 3$

So,
$$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

6

$$\frac{3-x}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$3 - x = A(x+5) + B(x+3)$

Substituting:

$x = -3: 6 = 2A$ so $A = 3$

$x = -5: 8 = -2B$ so $B = -4$

So,

$$\frac{3-x}{(x+3)(x+5)} = \frac{3}{x+3} - \frac{4}{x+5}$$

7

$$\frac{x+5}{(x+8)(x-4)} = \frac{A}{x+8} + \frac{B}{x-4} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$x + 5 = A(x-4) + B(x+8)$

Substituting:

$x = 4: 9 = 12B$ so $B = \frac{3}{4}$

$x = -8: -3 = -12A$ so $A = \frac{1}{4}$

$$\frac{x+5}{(x+8)(x-4)} = \frac{1}{4(x+8)} + \frac{3}{4(x-4)}$$

8

$$\frac{x-10}{2x(x+2)} = \frac{A}{2x} + \frac{B}{x+2} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$x - 10 = A(x+2) + B(2x)$

Substituting:

$x = 0: -10 = 2A$ so $A = -5$

$x = -2: -12 = -4B$ so $B = 3$

$$\frac{x-10}{2x(x+2)} = \frac{3}{x+2} - \frac{5}{2x}$$

9

$$\frac{2x-8}{(4x+5)(x+3)} = \frac{A}{(4x+5)} + \frac{B}{(x+3)} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$2x - 8 = A(x+3) + B(4x+5)$

Substituting:

$x = -\frac{5}{4}: -\frac{21}{2} = \frac{7}{4}A$ so $A = -6$

$x = -3: -14 = -7B$ so $B = 2$

$$\frac{2x-8}{(4x+5)(x+3)} = \frac{2}{(x+3)} - \frac{6}{(4x+5)}$$

10

$$\frac{7x-3}{(x+6)(x-3)} = \frac{A}{x+6} + \frac{B}{x-3} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$7x - 3 = A(x-3) + B(x+6)$

Substituting:

$x = -6: -45 = -9A$ so $A = 5$

$x = 3: 18 = 9B$ so $B = 2$

$$\frac{7x-3}{(x+6)(x-3)} = \frac{5}{x+6} + \frac{2}{x-3}$$

11

$$\frac{1}{(3x+1)(3x-1)} = \frac{A}{3x+1} + \frac{B}{3x-1} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$1 = A(3x-1) + B(3x+1)$

Substituting:

$x = -\frac{1}{3}: 1 = -2A$ so $A = -\frac{1}{2}$

$x = \frac{1}{3}: 1 = 2B$ so $B = \frac{1}{2}$

$$\frac{1}{(3x+1)(3x-1)} = \frac{1}{2(3x-1)} - \frac{1}{2(3x+1)}$$

12

$$\frac{8-x}{3x(x+4)} = \frac{A}{3x} + \frac{B}{x+4} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$8 - x = A(x+4) + B(3x)$

Substituting:

$x = 0: 8 = 4A$ so $A = 2$

$x = -4: 12 = -12B$ so $B = -1$

$$\frac{8-x}{3x(x+4)} = \frac{2}{3x} - \frac{1}{x+4}$$

13

$$\frac{3x-2a}{x(x-a)} = \frac{A}{x} + \frac{B}{x-a} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$3x - 2a = A(x-a) + B(x)$

Substituting:

$x = 0: -2a = -aA$ so $A = 2$

$x = a: a = aB$ so $B = 1$

$$\frac{3x-2a}{x(x-a)} = \frac{2}{x} + \frac{1}{x-a}$$

14

$$\frac{4}{(\sqrt{x}+1)(\sqrt{x}+3)} = \frac{A}{\sqrt{x}+1} + \frac{B}{\sqrt{x}+3}$$ for some constants A and B

Multiplying through by the denominator on the left:

$4 = A(\sqrt{x}+3) + B(\sqrt{x}+1)$

Tip: Since for real x we cannot let $\sqrt{x} < 0$ it seems unreasonable to substitute $\sqrt{x} = -1$ or -3; however, as you see in Chapter 4, if we allow x to take complex values then such a substitution is fine. This is a fine example of complex numbers allowing us to travel swiftly through calculations even though both the problem and solution involve only real values. Both substitution method and comparing coefficients are given below. Try using the substitution method both with the complex number option and show that the result is the same!

<u>Method 1: Substitution</u>

$x = 0: 4 = 3A + B \quad (1)$

$x = 1: 4 = 4A + 2B \quad (2)$

$2(1) - (2): 4 = 2A$

So $A = 2, B = -2$

$$\frac{4}{(\sqrt{x}+1)(\sqrt{x}+3)} = \frac{2}{\sqrt{x}+1} - \frac{2}{\sqrt{x}+3}$$

<u>Method 2: Comparing coefficients:</u>

$x^0: 4 = 3A + B \quad (1)$

$\sqrt{x}: 0 = A + B \quad (2)$

$(1) - (2): 4 = 2A$

So $A = 2, B = -2$

$$\frac{4}{(\sqrt{x}+1)(\sqrt{x}+3)} = \frac{2}{\sqrt{x}+1} - \frac{2}{\sqrt{x}+3}$$

15

$$\frac{1}{(x^2+2)(x^2+5)} = \frac{A}{x^2+2} + \frac{B}{x^2+5}$$ for some constants A and B

Multiplying through by the denominator on the left:

$1 = A(x^2+5) + B(x^2+2)$

Substituting:

$x = 0: 1 = 5A + 2B \quad (1)$

$x = 1: 1 = 6A + 3B \quad (2)$

$3(1) - 2(2): 1 = 3A$ so $A = \frac{1}{3}$

$(2): 3B = 1 - 6A = -1$ so $B = -\frac{1}{3}$

$$\frac{1}{(x^2+2)(x^2+5)} = \frac{1}{3(x^2+2)} - \frac{1}{3(x^2+5)}$$

16 **a**

$$\frac{4-5x}{(1+x)(2-x)} = \frac{A}{1+x} + \frac{B}{2-x} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$4 - 5x = A(2 - x) + B(1 + x)$

Substituting:

$x = -1: 9 = 3A$ so $A = 3$

$x = 2: -6 = 3B$ so $B = -2$

$$\frac{4-5x}{(1+x)(2-x)} = \frac{3}{1+x} - \frac{2}{2-x}$$

b

$$\frac{4-5x}{(1+x)(2-x)} = 3(1+x)^{-1} - \left(1 - \frac{x}{2}\right)^{-1}$$

$$= 3\left(1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \cdots\right)$$

$$- \left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \cdots\right)$$

$$= (3 - 3x + 3x^2 + \cdots) - \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots\right)$$

$$= 2 - \frac{7}{2}x + \frac{11}{4}x^2 + \cdots$$

c Expansion is valid where both $|x| < 1$ and $\left|-\frac{x}{2}\right| < 1$

So $|x| < 1$ and $|x| < 2$

The overlapping region of validity is $|x| < 1$

17 **a**

$$\frac{7x-2}{(1-x)(2+3x)} = \frac{A}{1-x} + \frac{B}{2+3x} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$7x - 2 = A(2 + 3x) + B(1 - x)$

Substituting:

$x = 1: 5 = 5A$ so $A = 1$

$x = -\frac{2}{3}: -\frac{20}{3} = \frac{5}{3}B$ so $B = -4$

$$\frac{7x-2}{(1-x)(2+3x)} = \frac{1}{1-x} - \frac{4}{2+3x}$$

b

$$\frac{7x-2}{(1-x)(2+3x)} = (1-x)^{-1} - 2\left(1 + \frac{3x}{2}\right)^{-1}$$

$$= \left(1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \cdots\right)$$

$$- 2\left(1 + (-1)\left(\frac{3x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2 + \cdots\right)$$

$$= (1 + x + x^2 + \cdots) - \left(2 - 3x + \frac{9}{2}x^2 + \cdots\right)$$
$$= -1 + 4x - \frac{7}{2}x^2 + \cdots$$

c Expansion is valid where both $|-x| < 1$ and $\left|\frac{3x}{2}\right| < 1$

So $|x| < 1$ and $|x| < \frac{2}{3}$

The overlapping region of validity is $|x| < \frac{2}{3}$

18

$$\frac{1}{2x^2 + 3x + 1} = \frac{1}{(2x + 1)(x + 1)} = \frac{A}{2x + 1} + \frac{B}{x + 1} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$1 = A(x + 1) + B(2x + 1)$

Substituting:

$x = -\frac{1}{2}: 1 = \frac{1}{2}A$ so $A = 2$

$x = -1: 1 = -B$ so $B = -1$

$$\frac{1}{2x^2 + 3x + 1} = \frac{2}{2x + 1} - \frac{1}{(x + 1)}$$

Applying binomial theorem to each partial fraction:

$$\frac{1}{2x^2 + 3x + 1} = 2(1 + 2x)^{-1} - (1 + x)^{-1}$$
$$= 2\left(1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \cdots\right)$$
$$- \left(1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \cdots\right)$$
$$= (2 - 4x + 8x^2 + \cdots) - (1 - x + x^2 + \cdots)$$
$$= 1 - 3x + 7x^2 + \cdots$$

b Expansion is valid where both $|2x| < 1$ and $|x| < 1$

So $|x| < \frac{1}{2}$ and $|x| < 1$

The overlapping region of validity is $|x| < \frac{1}{2}$

19

$$\frac{5x + 3}{1 + 2x - 3x^2} = \frac{5x + 3}{(1 + 3x)(1 - x)} = \frac{A}{1 + 3x} + \frac{B}{1 - x} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$5x + 3 = A(1 - x) + B(1 + 3x)$

Substituting:

$x = -\frac{1}{3}: \frac{4}{3} = \frac{4}{3}A$ so $A = 1$

$x = 1: 8 = 4B$ so $B = 2$

$$\frac{5x + 3}{1 + 2x - 3x^2} = \frac{1}{1 + 3x} + \frac{2}{1 - x}$$

Applying binomial theorem to each partial fraction:

$$\frac{5x + 3}{1 + 2x - 3x^2} = (1 + 3x)^{-1} + 2(1 - x)^{-1}$$

$$= \left(1 + (-1)(3x) + \frac{(-1)(-2)}{2!}(3x)^2 + \cdots\right)$$
$$+ 2\left(1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \cdots\right)$$
$$= (1 - 3x + 9x^2 + \cdots) + (2 + 2x + 2x^2 + \cdots)$$
$$= 3 - x + 11x^2 + \cdots$$

b Expansion is valid where both $|3x| < 1$ and $|-x| < 1$

So $|x| < \frac{1}{3}$ and $|x| < 1$

The overlapping region of validity is $|x| < \frac{1}{3}$

20

$$\frac{a}{x^2 - 3ax + 2a^2} = \frac{a}{(x-a)(x-2a)} = \frac{A}{x-a} + \frac{B}{x-2a}$$ for some constants A and B

Multiplying through by the denominator on the left:

$a = A(x - 2a) + B(x - a)$

Substituting:

$x = a$: $a = -aA$ so $A = -1$

$x = 2a$: $a = aB$ so $B = 1$

$$\frac{a}{x^2 - 3ax + 2a^2} = \frac{1}{x - 2a} - \frac{1}{x - a}$$

Exercise 2C

11

$$\begin{cases} 2x - 3y + 2z = 13 & (1) \\ 3x + y - z = 2 & (2) \\ 3x - 4y - 3z = 1 & (3) \end{cases}$$

$(1) + 2(2): 8x - y = 17 \qquad (4)$

$(3) - 3(2): -6x - 7y = -5 \qquad (5)$

$7(4) - (5): 62x = 124$

$x = 2$

$(4): y = 8x - 17 = -1$

$(2): z = 3x + y - 2 = 3$

12

$$\begin{cases} x + 4y - 5z = -3 & (1) \\ 2x - y + 5z = 12 & (2) \\ 8x + 5y + 11z = 30 & (3) \end{cases}$$

$(2) - 2(1): -9y + 15z = 18 \qquad (4)$

$(3) - 4(2): 9y - 9z = -18 \qquad (5)$

$(4) + (5): 6z = 0$

$z = 0$

$(5): 9y = -18 + 9z = -18$ so $y = -2$

$(1): x = 2z - 4y - 3 = 5$

13 **a**

$$\begin{cases} a + b + c = 4 & (1) \\ 9a + 3b + c = 14 & (2) \\ 16a + 4b + c = 25 & (3) \end{cases}$$

b From GDC: $a = 2, b = -3, c = 5$

14 **a**

$$\begin{cases} -a + b - c = 7 \\ 8a + 4b + 2c = 4 \\ 27a + 9b + 3c = 3 \end{cases}$$

b From GDC: $a = -1, b = 4, c = -2$

15 **a**

$$\begin{cases} x + 2y + kz = 8 & (1) \\ 2x + 5y + 2z = 7 & (2) \\ 5x + 12y + z = 2 & (3) \end{cases}$$

$(2) - 2(1): y + (2 - 2k)z = -9 \qquad (4)$

$(3) - 5(1): 2y + (1 - 5k)z = -38 \qquad (5)$

$(5) - 2(4): (-3 - k)z = -20 \qquad (6)$

When $k = -3$, this would be impossible and for any other value of k there would be a unique solution.

The system is inconsistent for $k = -3$.

b

Setting $k = 2$:

$(6): -5z = -20$ so $z = 4$

$(4): y = 2z - 9 = -1$

$(1): x = 8 - 2y - 2z = 2$

16 **a**

$$\begin{cases} kx + y + 2z = 4 & (1) \\ 3x + ky - 2z = 1 & (2) \\ -x + y + z = -2 & (3) \end{cases}$$

$(1) + (2): (k + 3)x + (k + 1)y = 5 \quad (4)$

$(2) + 2(3): x + (2 + k)y = -3 \quad (5)$

$(4) - (k + 3) \times (5): (k + 1)y - (k + 2)(k + 3)y = 5 + 3(k + 3)$

$(-k^2 - 4k - 5)y = 14 + 3k$

$$y = -\frac{14 + 3k}{k^2 + 4k + 5} = -\frac{14 + 3k}{1 + (k + 2)^2}$$

Since the denominator is a quadratic with no real roots (minimum value 1), y has a unique value for any value k. Therefore, the system is consistent for all $k \in \mathbb{R}$

b

Setting $k = 1$:

$$y = -\frac{17}{10} = -1.7$$

$(5): x = -3 - 3y = 2.1$

$(3): z = x - y - 2 = 1.8$

17 **a**

$$\begin{cases} x - y - 2z = 2 & (1) \\ 2x - 2y + z = 0 & (2) \\ 3x - 3y + 4z = a & (3) \end{cases}$$

$(2) - 2(1): 5z = -4$

$(3) - 3(1): 10z = a - 6$

For these to be consistent, $a - 6 = -8$ so $a = -2$

b

When $a = -2$, the system has solution $z = -0.8, x - y = 0.4$

Parameterising:

$x = 0.4 + \lambda, y = \lambda, z = -0.8$

18

$$\begin{cases} x - 2y + z = 2 & (1) \\ x + y - 3z = k & (2) \\ 2x - y - 2z = k^2 & (3) \end{cases}$$

$(2) - (1): 3y - 4z = k - 2$

$(3) - 2(1): 3y - 4z = k^2 - 4$

For the system to be consistent, $k - 2 = k^2 - 4 = (k - 2)(k + 2)$

So $k + 2 = 1$ or $k - 2 = 0$

$k = -1$ or 2

Then the system will have infinite solutions, since the three equations are not linearly independent (that is, any one can be produced as a linear combination of the other two).

19

Let the three numbers be x, y and z, where $x \le y \le z$

$\text{Mean} = \dfrac{x+y+z}{3}$

$\text{Range} = z - x$

$\text{Median} = y$

The information given can be summarised as:

$$\begin{cases} x+y+z=6y & (1) \\ z-x=5y & (2) \\ y-x=1 & (3) \end{cases}$$

Rearranging to standard format:

$$\begin{cases} x-5y+z=0 & (1) \\ x+5y-z=0 & (2) \\ x-y=-1 & (3) \end{cases}$$

$(1)+(2)\text{: } 2x=0$

$(3)\text{: } y=1$

$(1)\text{: } z=5$

The largest number is 5.

20

$$\begin{cases} -x+(2k-5)y-2z=3 & (1) \\ 3x-y+(k-1)z=4 & (2) \\ x+y+2z=-1 & (3) \end{cases}$$

$3(1)+(2)\text{: } (6k-16)y+(k-7)z=13 \quad (4)$

$(1)+(3)\text{: } (2k-4)y=2 \quad (5)$

From (5) if $k=2$ then the system is inconsistent

Otherwise:

$(k-2)\times(4)-(3k-8)(5)\text{: } (k-2)(k-7)z=7k-10$

The system is also inconsistent when $k=7$.

21

$$\begin{cases} 3x-y+5z=2 & (1) \\ 2x+4y+z=1 & (2) \\ x+y+kz=c & (3) \end{cases}$$

$(1)-3(3)\text{: } -4y+(5-3k)z=2-3c \quad (4)$

$(2)-2(3)\text{: } 2y+(1-2k)z=1-2c \quad (5)$

$(4)+2(5)\text{: } (7-7k)z=4-7c \quad (6)$

a If $k \ne 1$ then the system will have a unique solution

b If $k=1$ and $c=\frac{4}{7}$ then (6) reduces to $0=0$ and the system will have infinitely many solutions

If $z=2\lambda$ then (5) gives $2y-2\lambda=-\frac{1}{7}$ so $y=\lambda-\frac{1}{14}$

Then (3) gives $x=c-y-kz=\frac{4}{7}-\lambda+\frac{1}{14}-2\lambda=\frac{9}{14}-3\lambda$

Solution is $x = \frac{9}{14} - 3\lambda, y = \lambda - \frac{1}{14}, z = 2\lambda$

c If $k = 1$ and $c \neq \frac{4}{7}$ then there are no solutions, since (6) reduces to $0 \neq 0$

22

Let the three-digit number be written xyz so that its value is $100x + 10y + z$

Require that each of x, y and z be single digit integers.

The reversed number zyx has value $100z + 10y + x$

The information given can be summarised as:

$$\begin{cases} x + y + z = 16 & (1) \\ z = 2|x - y| & (2) \\ 100z + 10y + x = 100x + 10y + z - 297 & (3) \end{cases}$$

Rearranging to standard format and splitting the modulus statement into two cases:

$$\begin{cases} x + y + z = 16 & (1) \\ 2x - 2y + z = 0 & (2a) \\ 2x - 2y - z = 0 & (2b) \\ 99x - 99z = 297 & (3) \end{cases}$$

$(3): x - z = 3 \quad (4)$

$(1) + (4): 2x + y = 19 \quad (5)$

$(2a) + (4): 3x - 2y = 3 \; (6a)$

$(2b) - (4): x - 2y = -3 \; (6b)$

$2(5) + (6a): 7x = 41$ which does not have an integer solution

$2(3) + (6b): 5x = 35$ so $x = 7$

Then (4) gives $z = 4$ and (1) gives $y = 5$

The only solution is number 754.

Mixed Practice

1

$$(1-3x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(-3x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-3x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-3x)^3+\cdots$$
$$= 1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3+\cdots$$

2 **a**

$$(9+4x)^{-\frac{1}{2}} = \frac{1}{3}\left(1+\frac{4}{9}x\right)^{-\frac{1}{2}}$$
$$= \frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(\frac{4}{9}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{4}{9}x\right)^2+\cdots\right)$$
$$= \frac{1}{3}-\frac{2}{27}x+\frac{2}{81}x^2+\cdots$$

b Expansion is valid for $\left|\frac{4}{9}x\right| < 1$

That is $|x| < \frac{9}{4}$

3 **a**

$$\frac{1}{1+6x+9x^2} = \frac{1}{(1+3x)^2} = (1+3x)^{-2}$$

$a = 3, n = -2$

b

$$(1+3x)^{-2} = 1+(-2)(3x)+\frac{(-2)(-3)}{2!}(3x)^2+\frac{(-2)(-3)(-4)}{3!}(3x)^3+\cdots$$
$$= 1-6x+27x^2-108x^3+\cdots$$

4

$\frac{3x-1}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$ for some constants A and B

Multiplying through by the denominator on the left:

$3x-1 = A(x+1)+Bx$

Substituting:

$x = 0: -1 = A$

$x = -1: \ -4 = -B$ so $B = 4$

5

$\frac{5}{(3x-4)(x+2)} = \frac{A}{3x-4}+\frac{B}{x+2}$ for some constants A and B

Multiplying through by the denominator on the left:

$5 = A(x+2)+B(3x-4)$

Substituting:

$x = \frac{4}{3}: 5 = \frac{10}{3}A$ so $A = \frac{3}{2}$

$x = -2: 5 = -10B$ so $B = -\frac{1}{2}$

$$\frac{5}{(3x-4)(x+2)} = \frac{3}{2(3x-4)}-\frac{1}{2(x+2)}$$

6

$$\frac{27-x}{(x+6)(x-5)} = \frac{A}{x+6} + \frac{B}{x-5} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$27 - x = A(x-5) + B(x+6)$

Substituting:

$x = -6: 33 = -11A$ so $A = -3$

$x = 5: 22 = 11B$ so $B = 2$

$$\frac{27-x}{(x+6)(x-5)} = \frac{2}{x-5} - \frac{3}{x+6}$$

7

$$\begin{cases} 2x - y + 3z = 4 & (1) \\ 3x + 2y + 4z = 11 & (2) \\ 5x - 3y + 5z = -1 & (3) \end{cases}$$

$(2) + 2(1): 7x + 10z = 19 \quad (4)$

$3(1) - (3): x + 4z = 13 \quad (5)$

$7(5) - (4): 18z = 72$

So $z = 4$

$(5): x = 13 - 4z = -3$

$(1): y = 2x + 3z - 4 = 2$

8

$$\begin{cases} -2x + 3y + z = 4 & (1) \\ x - 4y + 2z = 8 & (2) \\ 7x - 18y + 4z = 16 & (3) \end{cases}$$

$(2) - 2(1): 5x - 10y = 0 \quad (4)$

$(3) - 4(1): 15x - 30y = 0 \quad (5)$

$(5) - 3(4): 0 = 0$

The system is consistent and has infinitely many solutions.

$(4): x = 2y$ so set $y = \lambda$ then $x = 2\lambda$ and from (1), $z = 4 + 2x - 3y = 4 + \lambda$

General solution: $x = 2\lambda, y = \lambda, z = 4\lambda$

9

$$\begin{cases} x + 3y + 4z = 2 & (1) \\ 3x + 8y + 12z = 5 & (2) \end{cases}$$

$3(1) - (2): y = 1$

Then (1) gives $x + 4z = -1$

Set $z = \lambda$.

General solution: $x = -1 - 4\lambda, y = 1, z = \lambda$

10 **a**

$$\begin{cases} 4a - 2b + c = 12 & (1) \\ a - b + c = 1 & (2) \\ a + b + c = -3 & (3) \end{cases}$$

b

$(1)-2(2): 2a-c=10 \ \ (4)$

$(1)+2(3): 6a+3c=6 \ \ (5)$

$3(4)+(5): 12a=36$ so $a=3$

$(4): c=2a-10=-4$

$(3): b=-3-a-c=-2$

11 **a**

$$\frac{1+2x}{(1+x)^2}=(1+2x)(1+x)^{-2}$$

$$=(1+2x)\left(1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\cdots\right)$$

$$=(1+2x)(1-2x+3x^2-4x^3+\cdots)$$

$$=1-x^2+2x^3+\cdots$$

b Expansion is valid for $|x|<1$

12

$$\frac{a+x}{(2-3x)^2}=\frac{(a+x)}{4}\left(1-\frac{3}{2}x\right)^{-2}$$

$$=\frac{(a+x)}{4}\left(1+(-2)\left(-\frac{3}{2}x\right)+\frac{(-2)(-3)}{2!}\left(-\frac{3}{2}x\right)^2+\cdots\right)$$

$$=\frac{(a+x)}{4}\left(1+3x+\frac{27}{4}x^2+\cdots\right)$$

The coefficient of x^2 is $\frac{a}{4}\times\frac{27}{4}+\frac{3}{4}=\frac{27a+12}{16}=\frac{15}{2}$

$27a+12=120$

$a=4$

13 **a**

$$(8+6x)^{\frac{2}{3}}=4\left(1+\frac{3}{4}x\right)^{\frac{2}{3}}$$

$$=4\left(1+\left(\frac{2}{3}\right)\left(\frac{3}{4}x\right)+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\left(\frac{3}{4}x\right)^2+\cdots\right)$$

$$=4\left(1+\frac{1}{2}x-\frac{1}{16}x^2+\cdots\right)$$

$$=4+2x-\frac{1}{4}x^2+\cdots$$

b Expansion is valid for $\left|\frac{3}{4}x\right|<1$

That is $|x|<\frac{4}{3}$

c

When $x = \frac{1}{3}$, which lies within the interval of validity for the expansion,

$(8 + 6x)^{\frac{2}{3}} = 10^{\frac{2}{3}} = \sqrt[3]{100}$

Using the expansion in part **a**:

$$\sqrt[3]{100} \approx 4 + \frac{2}{3} - \frac{1}{36}$$
$$\approx 4.639$$

14 **a**

$$\frac{10 - 3x}{(1 + 3x)(2 - 5x)} = \frac{A}{1 + 3x} + \frac{B}{2 - 5x} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:

$10 - 3x = A(2 - 5x) + B(1 + 3x)$

Substituting:

$x = -\frac{1}{3}: 11 = \frac{11}{3}A$ so $A = 3$

$x = \frac{2}{5}: \frac{44}{5} = \frac{11}{5}B$ so $B = 4$

$$\frac{10 - 3x}{(1 + 3x)(2 - 5x)} = \frac{3}{1 + 3x} + \frac{4}{2 - 5x}$$

b

$$\begin{aligned}\frac{10 - 3x}{(1 + 3x)(2 - 5x)} &= 3(1 + 3x)^{-1} + 2\left(1 - \frac{5}{2}x\right)^{-1}\\ &= 3\left(1 + (-1)(3x) + \frac{(-1)(-2)}{2!}(3x)^2 + \cdots\right)\\ &\quad + 2\left(1 + (-1)\left(-\frac{5}{2}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{5}{2}x\right)^2 + \cdots\right)\\ &= (3 - 9x + 27x^2 + \cdots) + \left(2 + 5x + \frac{25}{2}x^2 + \cdots\right)\\ &= 5 - 4x + \frac{79}{2}x^2 + \cdots\end{aligned}$$

c

Expansion is valid where both $|3x| < 1$ and $\left|-\frac{5}{2}x\right| < 1$

So $|x| < \frac{1}{3}$ and $|x| < \frac{2}{5}$

The overlapping region of validity is $|x| < \frac{1}{3}$

15

$$\begin{cases} 3x - y + z = 17 & (1)\\ x + 2y - z = 8 & (2)\\ 2x - 3y + 2z = k & (3)\end{cases}$$

$3(2) - (1): 7y - 4z = 7$ (4)

$2(2) - (3): 7y - 4z = 16 - k$ (5)

a For the system to be consistent, require $7 = 16 - k$ so $k = 9$

b Parameterizing the system: let $z = 7\lambda$
From (4): $7y = 7 + 4z = 7 + 28\lambda$ so $y = 1 + 4\lambda$
From (2): $x = 8 - 2y + z = 6 - \lambda$

16

$$\begin{cases} 2x - y + 3z = 2 & (1) \\ 3x + y + 2z = -2 & (2) \\ -x + 2y + az = b & (3) \end{cases}$$

$(1) + 2(3): 3y + (3 + 2a)z = 2 + 2b \quad (4)$
$(2) + 3(3): 7y + (2 + 3a)z = -2 + 3b \quad (5)$
$7(4) - 3(5): (15 + 5a)z = 20 + 5b$
$(3 + a)z = 4 + b (6)$
For the system to be consistent but have no unique solution, $a = -3$ and $b = -4$

17

$$\begin{aligned} \frac{1}{1 + x + x^2} &= \frac{1 - x}{1 - x^3} \\ &= (1 - x)(1 - x^3)^{-1} \\ &= (1 - x)\left(1 + (-1)(-x^3) + \frac{(-1)(-2)}{2!}(-x^3)^2 + \cdots\right) \\ &= (1 - x)(1 + x^3 + x^6 + \cdots) \\ &= 1 - x + x^3 + \cdots \end{aligned}$$

Tip: You could alternatively expand directly as

$$\begin{aligned} &\left(1 + (x + x^2)\right)^{-1} \\ &= 1 + (-1)(x + x^2) + \frac{(-1)(-2)}{2!}(x + x^2)^2 + \frac{(-1)(-2)(-3)}{3!}(x + x^2)^3 + \cdots \\ &= 1 - x - x^2 + (x^2 + 2x^3 + x^4) - (x^3 + 3x^4 + 3x^5 + x^6) + \cdots \\ &= 1 - x + x^3 + \cdots \end{aligned}$$

Although you will get the same terms, this method is problematic because it is far harder to establish an interval of validity, due to powers of $(x + x^2)$ impacting several coefficients of the final series – you may wish to read about the Ratio test for convergence of series to have an insight as to why this is important. Looking at the solution shown first, it is quickly clear that the interval of validity is $|x| < 1$.

18

Partial fractions:

$$\frac{11x - 3}{(2x - 1)(x - 3)} = \frac{A}{2x - 1} + \frac{B}{x - 3} \text{ for some constants } A \text{ and } B$$

Multiplying through by the denominator on the left:
$11x - 3 = A(x - 3) + B(2x - 1)$
Substituting:
$x = \frac{1}{2}: \frac{5}{2} = -\frac{5}{2}A$ so $A = -1$
$x = 3: 30 = 5B$ so $B = 6$

$$\frac{11x - 3}{(2x - 1)(x - 3)} = \frac{6}{x - 3} - \frac{1}{2x - 1}$$

$$= -2\left(1 - \frac{1}{3}x\right)^{-1} + (1 - 2x)^{-1}$$

$$= -2\left(1 + (-1)\left(-\frac{1}{3}x\right) + \frac{(-1)(-2)}{2!}\left(\frac{1}{3}x\right)^2 + \cdots\right)$$

$$+ \left(1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \cdots\right)$$

$$= -2 - \frac{2}{3}x - \frac{2}{9}x^2 + \cdots + 1 + 2x + 4x^2 + \cdots$$

$$= -1 + \frac{4}{3}x + \frac{34}{9}x^2 + \cdots$$

$$= -1 + \frac{4}{3}x + \cdots$$

19 **a**

$$\sqrt{\frac{1+5x}{1+12x}} = (1+5x)^{\frac{1}{2}}(1+12x)^{-\frac{1}{2}}$$

$$= \left(1 + \left(\frac{1}{2}\right)(5x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(5x)^2 + \cdots\right)\left(1 + \left(-\frac{1}{2}\right)(12x)\right.$$

$$\left. + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(12x)^2 + \cdots\right)$$

$$= \left(1 + \frac{5}{2}x - \frac{25}{8}x^2 + \cdots\right)(1 - 6x + 54x^2 + \cdots)$$

$$= 1 - \frac{7}{2}x + \cdots$$

b

Expansion is valid where both $|5x| < 1$ and $|12x| < 1$

So $|x| < \frac{1}{5}$ and $|x| < \frac{1}{12}$

The overlapping region of validity is $|x| < \frac{1}{12}$

c

When $x = 0.01$, $\sqrt{\frac{1+5x}{1+12x}} = \sqrt{\frac{1.05}{1.12}} = \sqrt{\frac{105}{112}} = \sqrt{\frac{7\times 15}{7\times 16}} = \frac{1}{4}\sqrt{15}$

Then $\sqrt{15} \approx 4(1 - 0.035)$

$\approx 4(0.965)$

≈ 3.86

20

$$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \cdots$$
$$= 1 - 3x + \frac{15}{2}x^2 + bx^3 + \cdots$$

Comparing coefficients:

$x^1: na = -3$ (1)

$x^2: \frac{n^2 - n}{2}a^2 = \frac{15}{2}$ (2)

$x^3: \frac{n(n-1)(n-2)}{3!}a^3 = b$ (3)

$(1)^2: n^2a^2 = 9$ (4)

$2(2): n^2a^2 - na^2 = 15$ (5)

$(4) - (5): na^2 = -6$ (6)

$(4)/(6): n = -\frac{3}{2}$

So $a = 2$

Then $(3): b = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{6}(2)^3 = -\frac{35}{2}$

21

$$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \cdots$$
$$= 1 - 9x + 54x^2 + bx^3 + \cdots$$

Comparing coefficients:

$x^1: na = -9$ (1)

$x^2: \frac{n^2 - n}{2}a^2 = 54$ (2)

$x^3: \frac{n(n-1)(n-2)}{3!}a^3 = b$ (3)

$(1)^2: n^2a^2 = 81$ (4)

$2(2): n^2a^2 - na^2 = 108$ (5)

$(4) - (5): na^2 = -27$ (6)

$(4)/(6): n = -3$

So $a = 3$

Then $(3): b = \frac{(-3)(-4)(-5)}{6}(3)^3 = -270$

22 **a**

$$\begin{cases} 2x + y + 6z = 0 & (1) \\ 4x + 3y + 14z = 4 & (2) \\ 2x - 2y + (\alpha - 2)z = \beta - 12 & (3) \end{cases}$$

$2(1) - (2): -y - 2z = -4 \quad (4)$

$(1) - (3): 3y + (8 - \alpha)z = 12 - \beta \quad (5)$

$3(4) + (5): (2 - \alpha)z = -\beta$

i If $\alpha = 2$ and $\beta \neq 0$ then the system is inconsistent (has no solutions)

ii If $\alpha \neq 2$ then the system has a unique solution

iii If $\alpha = 2$ and $\beta = 0$ then the system is consistent and has infinitely many solutions.

b When $\alpha = 2$ and $\beta = 0$, (4) and (5) have the same information.

(4) gives $y = 4 - 2z$

(1) gives $2x = -y - 6z = -4 - 4z$ so $x = -2 - 2z$

In Cartesian form, this can be expressed as

$$\frac{x+2}{-2} = \frac{y-4}{-2} = z$$

The parameterised version would be $x = -2 - 2\lambda, y = 4 - 2\lambda, z = \lambda$

3 Trigonometry

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 3A

8 $\text{cosec}\, A = \frac{5}{3}, \sec B = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

9
Taking the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$ $(*)$

$$\sin^2\theta + \cot^2\theta \sin^2\theta = \sin^2\theta + \frac{\cos^2\theta}{\sin^2\theta} \times \sin^2\theta \quad \text{by definition of } \cot\theta$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1 \quad \text{by } (*)$$

10 From GDC, the only solution is $x = 0.644$

11 **a** From GDC: The only stationary point is a local minimum at $(0.715, 2.39)$

b There is no upper bound to the values taken by $f(x)$ so the range is $f(x) \geq 2.39$

12 Using the definition of $\cot A$,

$$\sin A \cot A = \sin A \times \frac{\cos A}{\sin A}$$
$$= \cos A$$

13 Using the definition of $\tan B$, $\sec B$ and $\text{cosec}\, B$,

$$\tan B\, \text{cosec}\, B = \frac{\sin B}{\cos B} \times \frac{1}{\sin B}$$
$$= \frac{1}{\cos B}$$
$$= \sec B$$

14 $\arcsin(\sin\pi) = \arcsin 0 = 0$

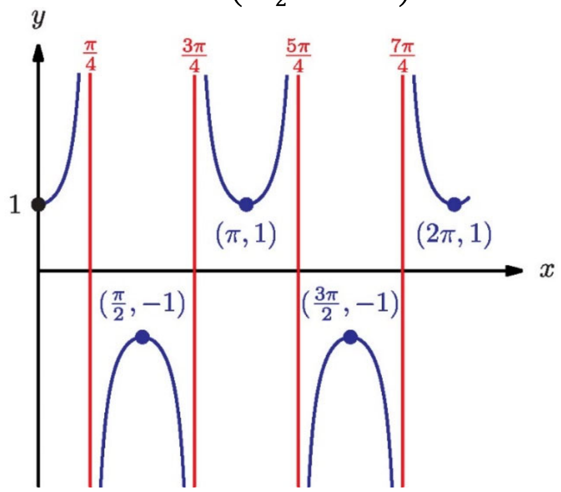
15

$$y = \frac{1}{\cos 2x}$$

Vertical asymptotes where $\cos 2x = 0$: $x = \frac{(2n+1)}{4}\pi$

Local minima at $(n\pi, 1)$

Local maxima at $\left(\frac{(2n+1)}{2}\pi, -1\right)$

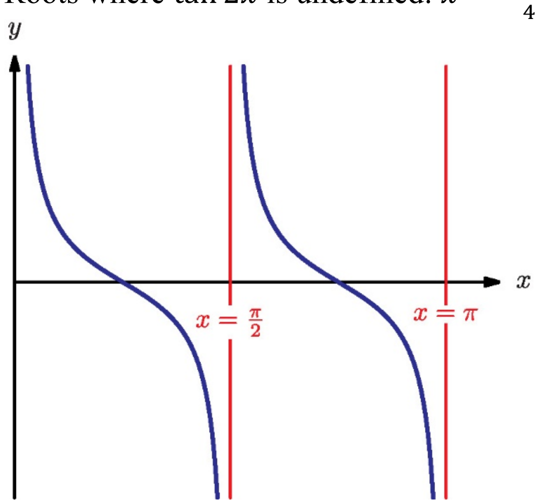


16

$$y = \frac{3}{\tan 2x}$$

Vertical asymptotes where $\tan 2x = 0$: $x = \frac{n}{2}\pi$

Roots where $\tan 2x$ is undefined: $x = \frac{2n+1}{4}\pi$



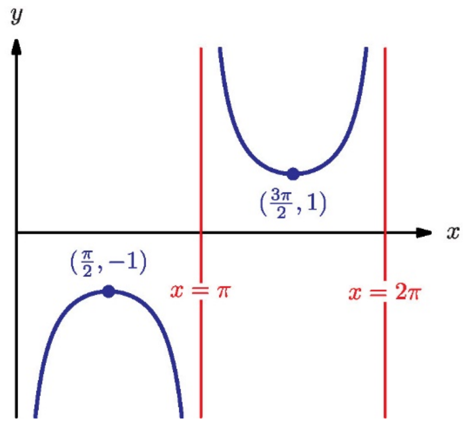
17

$$y = \frac{1}{\sin(x - \pi)} = -\frac{1}{\sin x}$$

Vertical asymptotes where $\sin x = 0$: $x = n\pi$

Local maxima at $\left(\left(2n + \frac{1}{2}\right)\pi, -1\right)$

Local minima at $\left(\left(2n - \frac{1}{2}\right)\pi, 1\right)$



18

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$
$$= \frac{1}{\cos x \sin x}$$
$$= \sec x \operatorname{cosec} x$$

19

$$\sec x - \cos x \equiv \frac{1}{\cos x} - \cos x$$
$$\equiv \frac{1 - \cos^2 x}{\cos x}$$
$$\equiv \frac{\sin^2 x}{\cos x}$$
$$\equiv \sin x \times \frac{\sin x}{\cos x}$$
$$\equiv \sin x \tan x$$

20

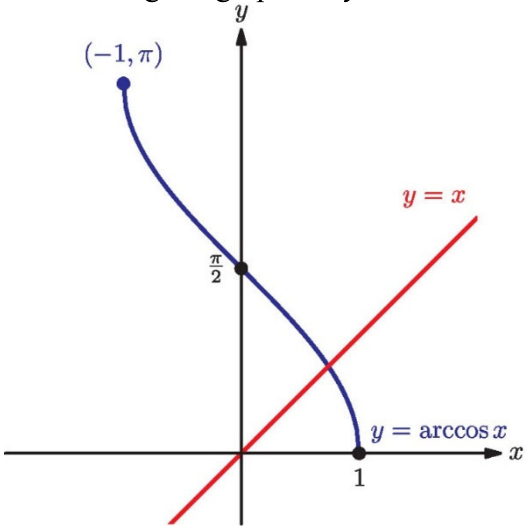
$$\frac{\sin\theta}{1 - \cos\theta} - \frac{\sin\theta}{1 + \cos\theta} = \sin\theta \frac{(1 + \cos\theta) - (1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)}$$
$$= \sin\theta \times \frac{2\cos\theta}{1 - \cos^2\theta}$$
$$= \sin\theta \times \frac{2\cos\theta}{\sin^2\theta}$$
$$= \frac{2\cos\theta}{\sin\theta}$$
$$= 2\cot\theta$$

21

$2\tan^2 x + 3\sec x = 0$
$2(\sec^2 x - 1) + 3\sec x = 0$
$2\sec^2 x + 3\sec x - 2 = 0$
$(\sec x + 2)(2\sec x - 1) = 0$
$\sec x = -2$ or $\frac{1}{2}$ (reject; outside the range of $\sec x$)
$\cos x = -\frac{1}{2}$
$x = \pm\frac{2\pi}{3}$

22

Considering the graphs of $y = \arccos x$ and $y = 2x$:



$\arccos x = 2x$ has only one solution.

23 $\arccos(-x) = \pi - \arccos x$

24 **a**

Using the identity $1 + \tan^2\theta = \sec^2\theta$:
$\sec^2 x - 3\tan x + 1 = (1 + \tan^2 x) - 3\tan x + 1$
$= \tan^2 x - 3\tan x + 2$
So if $\sec^2 x - 3\tan x + 1 = 0$, then $\tan^2 x - 3\tan x + 2 = 0$

b

$(\tan x - 1)(\tan x - 2) = 0$
$\tan x = 1$ or 2

c

Primary solutions: $x_1 = \arctan 1 = \frac{\pi}{4}$ and $x_2 = \arctan 2 = 1.11$
Secondary solutions: $x_3 = x_1 + \pi = \frac{5\pi}{4}$ and $x_4 = x_2 + \pi = 4.25$

25

$$\operatorname{cosec} 2x = \frac{1}{\sin 2x}$$
$$= \frac{1}{2\sin x\cos x}$$
$$= \frac{1}{2}\operatorname{cosec} x \sec x$$

26

$$\sec 2\theta = \frac{1}{\cos 2\theta}$$
$$= \frac{1}{2\cos^2\theta - 1}$$

Dividing numerator and denominator by $\cos^2\theta$:

$$\sec 2\theta = \frac{\sec^2\theta}{2 - \sec^2\theta}$$

27

Let $y = \sec x$ so that $x = \operatorname{arcsec} y$

Then $\frac{1}{y} = \cos x$

$$x = \arccos\left(\frac{1}{y}\right)$$

$$\operatorname{arcsec} y = \arccos\left(\frac{1}{y}\right)$$

Changing variables:

$$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$$

28 **a**

At P, $y = 0$ so $4x\cos\theta = 20$: $x = 5\sec\theta$

At Q, $x = 0$ so $5y\sin\theta = 20$: $y = 4\operatorname{cosec}\theta$

P: $(5\sec\theta, 0)$; Q: $(0, 4\operatorname{cosec}\theta)$

Then M has coordinates $(2.5\sec\theta, 2\operatorname{cosec}\theta)$

b

Substituting $x = 2.5\sec\theta$ and $y = 2\operatorname{cosec}\theta$:

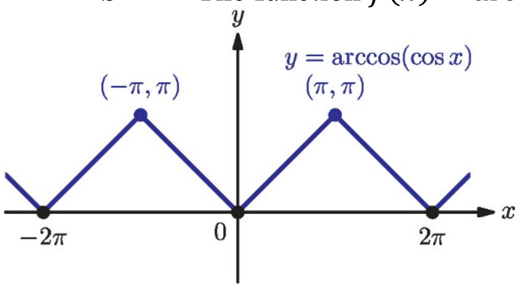
$$\frac{25}{x^2} + \frac{16}{y^2} = 4\cos^2\theta + 4\sin^2\theta = 4$$

So M does lie on the curve.

29 **a** Domain: $x \in \mathbb{R}$

Range: $0 \le f(x) \le \pi$

b The function $f(x) = \arccos(\cos x)$ repeats every 2π:

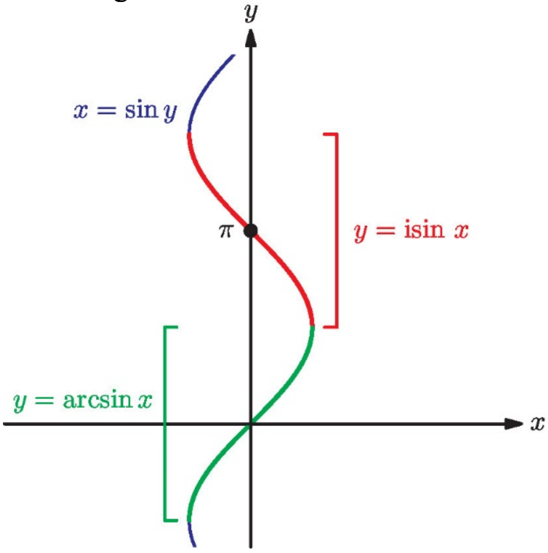


i $f(x) = x - 2\pi$

ii $f(x) = 2\pi - x$

iii $f(x) = -x$

30
To see the graph of $y = \text{isin}\, x$, draw the graph of $x = \sin y$ and specify the values of y according to the definition:



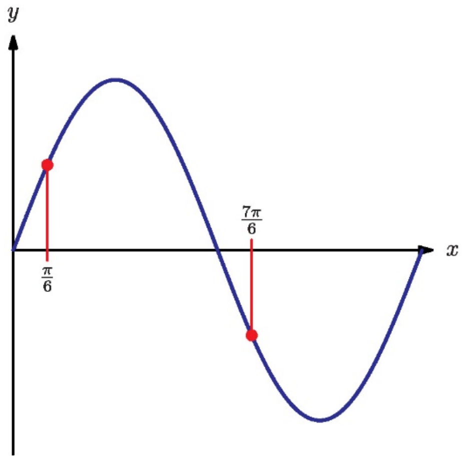
If the green part of the curve ($y = \arcsin x$) represents the interval of primary solutions to a problem $\sin y = k$ then the red part of the curve ($y = \text{isin}\, x$) is equivalent to the secondary solutions. Since a secondary solution y_2 is found from the primary solution y_1 by $y_2 = \pi - y_1$, it follows that:
$\text{isin}\, x = \pi - \arcsin x$

Exercise 3B

11 **a**

$$\sin(\pi + x) \equiv \sin \pi \cos x + \cos \pi \sin x$$
$$\equiv 0 \cos x - \sin x$$
$$\equiv -\sin x$$

b



12

$$\sin\left(\theta + \frac{\pi}{3}\right) + \sin\left(\theta - \frac{\pi}{3}\right)$$
$$\equiv \left(\sin\theta\cos\left(\frac{\pi}{3}\right) + \cos\theta\sin\left(\frac{\pi}{3}\right)\right)$$
$$+ \left(\sin\theta\cos\left(-\frac{\pi}{3}\right) + \cos\theta\sin\left(-\frac{\pi}{3}\right)\right)$$
$$\equiv \left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) + \left(\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right)$$
$$\equiv \sin\theta$$

13

$$\cos x + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(x + \frac{4\pi}{3}\right)$$
$$\equiv \cos x + \left(\cos x\cos\left(\frac{2\pi}{3}\right) - \sin x\sin\left(\frac{2\pi}{3}\right)\right)$$
$$+ \left(\cos x\cos\left(\frac{4\pi}{3}\right) - \sin x\sin\left(\frac{4\pi}{3}\right)\right)$$
$$\equiv \cos x + \left(-\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$$
$$+ \left(-\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)$$
$$\equiv 0$$

14 **a**

$$\sin(x + 45°) + \cos(x + 45°)$$
$$\equiv (\sin x\cos 45° + \cos x\sin 45°) + (\cos x\cos 45° - \sin x\sin 45°)$$
$$\equiv \left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) + \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$
$$\equiv \sqrt{2}\cos x$$

b

$$\sqrt{2}\cos x = \frac{\sqrt{2}}{2}$$
$$\cos x = \frac{1}{2}$$
$$x = 60° \text{ or } 300°$$

15

$$\cot x - \tan x \equiv \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$
$$\equiv \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}$$
$$\equiv \frac{\cos 2x}{\frac{1}{2}\sin 2x}$$
$$\equiv 2\cot 2x$$

16

$$\tan\left(\theta + \frac{\pi}{4}\right) - \tan\left(\theta - \frac{\pi}{4}\right) \equiv \frac{\tan\theta + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\theta\tan\left(\frac{\pi}{4}\right)} - \frac{\tan\theta - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\theta\tan\left(\frac{\pi}{4}\right)}$$
$$\equiv \frac{\tan\theta + 1}{1 - \tan\theta} - \frac{\tan\theta - 1}{1 + \tan\theta}$$
$$\equiv \frac{(\tan\theta + 1)^2 + (\tan\theta - 1)^2}{(1 - \tan\theta)(1 + \tan\theta)}$$
$$\equiv \frac{(\tan^2\theta + 2\tan\theta + 1) + (\tan^2\theta - 2\tan\theta + 1)}{1 - \tan^2\theta}$$
$$\equiv \frac{2\tan^2\theta + 2}{1 - \tan^2\theta}$$

17 **a**

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
$$= \frac{\frac{2}{3}}{1 - \frac{1}{9}}$$
$$= \frac{\left(\frac{6}{9}\right)}{\left(\frac{8}{9}\right)}$$
$$= \frac{3}{4}$$

b

$$\tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$$
$$= \frac{\frac{1}{3} + \frac{3}{4}}{1 - \frac{1}{3} \times \frac{3}{4}}$$
$$= \frac{\left(\frac{13}{12}\right)}{\left(\frac{3}{4}\right)}$$

$$= \frac{13}{9}$$

18

$$\tan(\theta - 45°) = \frac{1}{2} = \frac{\tan\theta - \tan 45°}{1 + \tan\theta \tan 45°}$$
$$= \frac{\tan\theta - 1}{1 + \tan\theta}$$

$$\tan\theta - 1 = \frac{1}{2}(1 + \tan\theta)$$

$$\frac{1}{2}\tan\theta = \frac{3}{2}$$

$$\tan\theta = 3$$

19 **a**

$$0 < A < \frac{\pi}{2} \text{ so } 0 < \cos A < 1$$

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

b

$$0 < B < \frac{\pi}{2} \text{ so } 0 < \cos B < 1$$

$$\cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$= \frac{2\sqrt{2}}{3} \times \frac{3}{5} - \frac{1}{3} \times \frac{4}{5}$$
$$= \frac{6\sqrt{2} - 4}{15}$$

20

x and y are both acute, so $0 < \cos x, \cos y < 1$

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13}$$
$$= \frac{16}{65}$$

21

$$\sin(\theta + 60°) = 2\cos\theta$$

$$\sin\theta \cos 60° + \cos\theta \sin 60° = 2\cos\theta$$

$$\frac{1}{2}\sin\theta = \left(2 - \frac{\sqrt{3}}{2}\right)\cos\theta$$

$$\tan\theta = 4 - \sqrt{3}$$

Worked solutions

22

$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y} = 2$$
$$\tan x - \tan y = 2 + 2\tan x \tan y$$
$$\tan y\,(2\tan x + 1) = \tan x - 2$$
$$\tan y = \frac{\tan x - 2}{2\tan x + 1}$$

Tip: Alternatively:

$x - y = \arctan 2$

$y = x - \arctan 2$

$$\tan y = \tan(x - \arctan 2)$$
$$= \frac{\tan x - \tan(\arctan 2)}{1 + \tan x \tan(\arctan 2)}$$
$$= \frac{\tan x - 2}{1 + 2\tan x}$$

23

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A} = -\frac{3}{4}$$
$$8\tan A = -3 + 3\tan^2 A$$
$$3\tan^2 A - 8\tan A - 3 = 0$$
$$(3\tan A + 1)(\tan A - 3) = 0$$
$$\tan A = -\frac{1}{3} \text{ or } 3$$

24

$$\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{24}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{24}\right) \equiv \cos\left(\frac{\pi}{8}+\frac{\pi}{24}\right)$$
$$\equiv \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$

25

$\cos 15° + \sqrt{3}\sin 15° = R\sin x \cos 15° + R\cos x \sin 15° \equiv R\sin(x + 15°)$

where $R\sin x = 1$ and $R\cos x = \sqrt{3}$

$$R = \sqrt{R^2(\sin^2 x + \cos^2 x)}$$
$$= \sqrt{1^2 + \sqrt{3}^2}$$
$$= 2$$

$$\tan x \equiv \frac{R\sin x}{R\cos x} = \frac{1}{\sqrt{3}} \text{ so } x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

Then $\cos 15° + \sqrt{3}\sin 15° = 2\sin(45°) = \sqrt{2}$

26

$$\cot(A+B) \equiv \frac{\cos(A+B)}{\sin(A+B)}$$
$$\equiv \frac{\cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B}$$

Dividing numerator and denominator by $\sin A\sin B$:

$$\cot(A+B) \equiv \frac{\cot A\cot B - 1}{\cot A + \cot B}$$

27 **a** $\sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$

b

$\sin 2x = \tan x$

$2\sin x\cos x = \dfrac{\sin x}{\cos x}$

$\dfrac{\sin x}{\cos x}(2\cos^2 x - 1) = 0$

$\sin x = 0$ or $\cos x = \pm\dfrac{1}{\sqrt{2}}$

Solutions for $0 \le x \le 2\pi$:

$x = 0, \pi, 2\pi$ or $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

28

Let $t = \tan 22.5°$

Then using double angle identity for tan, $\tan 45° = \dfrac{2t}{1-t^2}$

$\dfrac{2t}{1-t^2} = 1$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$

$t = -1 \pm \sqrt{2}$

Since $\tan 22.5°$ must be positive, $\tan 22.5° = \sqrt{2} - 1$

29 **a**

$$\tan\left(x - \frac{\pi}{4}\right) \equiv \frac{\tan x - \tan\left(\frac{\pi}{4}\right)}{1 + \tan x\tan\left(\frac{\pi}{4}\right)}$$
$$\equiv \frac{\tan x - 1}{\tan x + 1}$$

b

$\dfrac{\tan\theta - 1}{\tan\theta + 1} = 6\tan\theta$

$6\tan^2\theta + 6\tan\theta = \tan\theta - 1$

$6\tan^2\theta + 5\tan\theta + 1 = 0$

$(3\tan\theta + 1)(2\tan\theta + 1) = 0$

$\tan\theta = -\dfrac{1}{3}$ or $-\dfrac{1}{2}$

30 **a**

$\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Then $\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta\,(1 - \sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

b

$\sin 3x = 2\sin x$

$3\sin x - 4\sin^3 x = 2\sin x$

$\sin x\,(1 - 4\sin^2 x) = 0$

$\sin x = 0$ or $\sin x = \pm\frac{1}{2}$

Solutions for $0 \le x \le \pi$:

$x = 0, \pi$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}$

31 **a**

$\sin(\pi - x) = \sin\pi\cos x - \cos\pi\sin x$

$= 0\cos x + \sin x$

$= \sin x$

b

Angle in a triangle sum to π radians so $C = \pi - 2A$

From part **a**:

$\sin C = \sin 2A \equiv 2\sin A\cos A$

Then $\frac{\sin C}{\sin A} = 2\cos A$

32 **a**

$3\sin x - 7\cos x = R\sin x\cos\theta - R\cos x\sin\theta \equiv R\sin(x - \theta)$

where $R\cos\theta = 3$ and $R\sin\theta = 7$

$R = \sqrt{R^2(\sin^2\theta + \cos^2\theta)}$

$= \sqrt{7^2 + 3^2}$

$= \sqrt{58}$

b

Then the minimum value of $10 + 3\sin x - 7\cos x$ is $10 - \sqrt{58}$

So the maximum value of the expression is $\frac{3}{10 - \sqrt{58}}$

33 **a**

$3\sin\theta + \sqrt{3}\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha \equiv R\sin(\theta + \alpha)$

where $R\cos\alpha = 3$ and $R\sin\alpha = \sqrt{3}$

$R = \sqrt{R^2(\sin^2\theta + \cos^2\theta)}$

$= \sqrt{\sqrt{3}^2 + 3^2}$

$= 2\sqrt{3}$

$\tan\alpha \equiv \frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3}$ so $\alpha = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$3\sin\theta + \sqrt{3}\cos\theta = 2\sqrt{3}\sin\left(\theta + \frac{\pi}{6}\right)$

b

The maximum value of $6 + 3\sin\theta + \sqrt{3}\cos\theta$ is therefore $6 + 2\sqrt{3}$

So the minimum of $f(x)$ is

$$\frac{1}{6+2\sqrt{3}} = \frac{1}{6+2\sqrt{3}} \times \frac{6-2\sqrt{3}}{6-2\sqrt{3}} = \frac{6-2\sqrt{3}}{36-12} = \frac{3-\sqrt{3}}{12}$$

This occurs when $2\sqrt{3}\sin\left(2x + \frac{\pi}{6}\right) = 2\sqrt{3}$ so $2x = \frac{\pi}{3}$

$$x = \frac{\pi}{6}$$

34

Using the compound angle formula for $\tan(A+B)$ and that $\tan(\arctan p) = p$:

$$\tan\left(\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{5}\right)\right) = \frac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{2}\times\frac{1}{5}}$$

$$= \frac{\frac{3}{10}}{\frac{11}{10}}$$

$$= \frac{3}{11}$$

35 **a**

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x$$

$$= 1\cos x - 0\sin x$$

$$= \cos x$$

b

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Using part **a** to rewrite the cosine in terms of sine:

$$\cos(A+B) = \sin\left(\frac{\pi}{2} - A - B\right)$$

$$= \sin\left(\left(\frac{\pi}{2} - A\right) - B\right)$$

Using the compound angle formula for sine:

$$\cos(A+B) = \sin\left(\frac{\pi}{2} - A\right)\cos B - \cos\left(\frac{\pi}{2} - A\right)\sin B$$

Then using that $\sin\left(\frac{\pi}{2} - A\right) = \cos A$ and therefore also $\cos\left(\frac{\pi}{2} - A\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - A\right)\right) = \sin A$:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

36 **a**

$$\tan 3\theta \equiv \tan(\theta + 2\theta)$$

$$\equiv \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$$

$$\equiv \frac{\tan\theta + \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}{1 - \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}$$

$$\equiv \frac{\left(\frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}\right)}{\left(\frac{1-3\tan^2\theta}{1-\tan^2\theta}\right)}$$

$$\equiv \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

b

Since $\tan 30° = \frac{1}{\sqrt{3}}$

If $\tan 10° = t$ then by part **a**,

$$\frac{1}{\sqrt{3}} = \frac{3t - t^3}{1-3t^2}$$

So $\frac{1}{\sqrt{3}} - \sqrt{3}t^2 = 3t - t^3$

$t^3 - \sqrt{3}t^2 - 3t + \frac{\sqrt{3}}{3} = 0$, as required.

Mixed Practice

1

$\sec\theta = 3$ so $\cos\theta = \frac{1}{3}$

$\cos 2\theta = 2\cos^2\theta - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$

2 **a**

$$\cos\left(x+\frac{\pi}{4}\right) + \cos\left(x-\frac{\pi}{4}\right) \equiv \left(\cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4}\right) + \left(\cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4}\right)$$

$$\equiv \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) + \left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$\equiv \sqrt{2}\cos x$$

b

$\sqrt{2}\cos x = \sqrt{2}\sin x$

$\tan x = 1$

Solutions for $0 \le x < \pi$:

$x = \frac{\pi}{4}$

3 **a**

$$\sin\left(x+\frac{\pi}{3}\right) + \cos\left(x+\frac{\pi}{6}\right) \equiv \left(\sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3}\right) + \left(\cos x\cos\frac{\pi}{6} - \sin x\sin\frac{\pi}{6}\right)$$

$$\equiv \left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right) + \left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$\equiv \sqrt{3}\cos x$$

b

$\sqrt{3}\cos x = \sin x$

$\tan x = \sqrt{3}$

Solutions for $0 < x < 2\pi$:

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

4 **a**

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{6}{-8} = -\frac{3}{4}$$

b

Angle is acute, so $0 < \cos\theta < 1$ and so $\sec\theta > 1$

$\sec^2\theta = \tan^2\theta + 1 = 10$

Selecting the positive root: $\sec\theta = \sqrt{10}$

5

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= \cos A\cos B - \sqrt{\sin^2 A\sin^2 B}\\ &= \cos A\cos B - \sqrt{(1-\cos^2 A)(1-\cos^2 B)}\\ &= \frac{1}{2}\times\frac{1}{3} \pm \sqrt{\frac{3}{4}\times\frac{8}{9}}\\ &= \frac{1}{6} \pm \sqrt{\frac{2}{3}}\\ &= \frac{1\pm 2\sqrt{6}}{6}\end{aligned}$$

6

$(\arcsin x)^2 = \frac{\pi^2}{9}$

$\arcsin x = \pm\frac{\pi}{3}$

$x = \sin\left(\pm\frac{\pi}{3}\right)$

$\quad = \pm\frac{\sqrt{3}}{2}$

7

$$\begin{aligned}\tan 105° &= \tan(60° + 45°)\\ &= \frac{\tan 60° + \tan 45°}{1 - \tan 60°\tan 45°}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}}\\ &= \frac{(\sqrt{3}+1)^2}{(1-\sqrt{3})(1+\sqrt{3})}\\ &= \frac{4+2\sqrt{3}}{-2}\\ &= -2-\sqrt{3}\end{aligned}$$

8

$\sin\left(x+\frac{\pi}{3}\right) = \sin\left(x-\frac{\pi}{3}\right)$

$\sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} = \sin x\cos\frac{\pi}{3} - \cos x\sin\frac{\pi}{3}$

$2\cos x\sin\frac{\pi}{3} = 0$

$\cos x = 0$

Solutions for $0 < x < 2\pi$: $x = \frac{\pi}{2}$

9

$\tan x + \tan 2x = 0$

$\tan x + \frac{2\tan x}{1-\tan^2 x} = 0$

$\frac{\tan x}{1-\tan^2 x}(3-\tan^2 x) = 0$

$\tan x = 0$ or $\tan x = \pm\sqrt{3}$

Solutions for $0° \le x < 360°$:

$x = 180°$ or $60°, 120°, 240°, 300°$

10 **a**

$\arctan\frac{1}{2} - \arctan\frac{1}{3} = \arctan a, a \in \mathbb{Q}^+$

Taking tan of both sides:

$\tan\left(\arctan\frac{1}{2} - \arctan\frac{1}{3}\right) = \tan(arctan\ a)$

Using compound angle formula for $\tan(A-B)$ and $\tan(\arctan x) = x$:

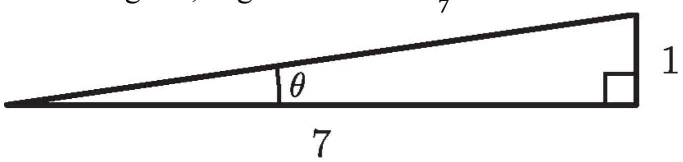
$$\frac{\frac{1}{2}-\frac{1}{3}}{1+\frac{1}{2}\times\frac{1}{3}} = a$$

$$a = \frac{\left(\frac{1}{6}\right)}{\left(\frac{7}{6}\right)} = \frac{1}{7}$$

b

$\arcsin x = \arctan\frac{1}{7}$

In the diagram, angle $\theta = \arctan\frac{1}{7}$



By Pythagoras Theorem, the hypotenuse has length $\sqrt{1^2+7^2} = \sqrt{50}$

$x = \sin\left(arctan\frac{1}{7}\right) = \frac{1}{\sqrt{50}}$

11 **a**

$\sin 2\theta \equiv 2\sin\theta\cos\theta$

$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$\tan 2\theta \equiv \frac{\sin 2\theta}{\cos 2\theta}$

$\equiv \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

Dividing through by $\cos^2\theta$:

$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$

b

$\tan 225° = 1$

Using part **a**, if $\tan 112.5° = t$

$1 = \frac{2t}{1-t^2}$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$

$t = -1 \pm \sqrt{2}$

Since $90° < 112.5° < 180°$, $\tan 112.5° < 0$

So $\tan 112.5° = -1 - \sqrt{2}$

12 Using compound angle formulae:

$\sin x \cos\left(\frac{\pi}{6}\right) + \cos x \sin\left(\frac{\pi}{6}\right) + \sin x \cos\left(-\frac{\pi}{6}\right) + \cos x \sin\left(-\frac{\pi}{6}\right) = 3\cos x$

$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = 3\cos x$

$\sqrt{3}\sin x = 3\cos x$

$\tan x = \sqrt{3}$

The only solution in the given interval is $x = \frac{\pi}{3}$

13

$\cos y = \sin(x+y) \equiv \sin x \cos y + \cos x \sin y$

$\cos y\,(1 - \sin x) = \cos x \sin y$

Then $\frac{\sin y}{\cos y} \equiv \tan y = \frac{1-\sin x}{\cos x} = \sec x - \tan x$

14

$\cot 2x \equiv \frac{\cos 2x}{\sin 2x}$

$\equiv \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x}$

Dividing numerator and denominator by $\sin^2 x$:

$\cot 2x \equiv \frac{\cot^2 x - 1}{2\cot x}$

15 **a**

$\operatorname{cosec} 2x - \cot 2x \equiv \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}$

$\equiv \frac{1-\cos 2x}{\sin 2x}$

$\equiv \frac{2\sin^2 x}{2\sin x \cos x}$

$\equiv \frac{\sin x}{\cos x}$

$\equiv \tan x$

b

$$\tan\left(\frac{3\pi}{8}\right) = \operatorname{cosec}\left(\frac{3\pi}{4}\right) - \cot\left(\frac{3\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} - (-1)$$
$$= 1 + \frac{\sqrt{2}}{2}$$

16 **ai**

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$
$$= 1 \times \cos 2\theta$$
$$= \cos 2\theta$$

aii

$$\sin^2 2\theta\,(\cot^2\theta - \tan^2\theta) \equiv (2\sin\theta\cos\theta)^2\left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right)$$
$$\equiv 4\sin^2\theta\cos^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right)$$
$$\equiv 4(\cos^4\theta - \sin^4\theta)$$

b

$$4\cos 2\theta = 2$$
$$\cos 2\theta = \frac{1}{2}$$

Solutions for $0 \le \theta < 2\pi$:

$$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

17 **a**

$$3\sin x + \sqrt{3}\cos x = R\sin x\cos\theta + R\cos x\sin\theta \equiv R\sin(x+\theta)$$

where $R\cos\theta = 3$ and $R\sin\theta = \sqrt{3}$

$$R = \sqrt{R^2(\sin^2\theta + \cos^2\theta)}$$
$$= \sqrt{\sqrt{3}^2 + 3^2}$$
$$= 2\sqrt{3}$$

$$\frac{R\sin\theta}{R\cos\theta} \equiv \tan\theta = \frac{\sqrt{3}}{3}$$

So $\theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

$$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$$

b

$$2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right) = 3$$
$$\sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Solutions for $-\pi < x < \pi$:

$x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{2\pi}{3}$

$x = \frac{\pi}{6}, \frac{\pi}{2}$

18

$\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$

$\sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3} = 2\sin x \sin\left(\frac{\pi}{3}\right)$

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 2\sin x \times \frac{\sqrt{3}}{2}$

$\sqrt{3}\cos x = \left(2\sqrt{3} - 1\right)\sin x$

$$\begin{aligned}\tan x &= \frac{\sqrt{3}}{2\sqrt{3} - 1}\\ &= \frac{\sqrt{3}\left(2\sqrt{3} + 1\right)}{\left(2\sqrt{3} - 1\right)\left(2\sqrt{3} + 1\right)}\\ &= \frac{6 + \sqrt{3}}{11}\end{aligned}$$

So $11\tan x = 6 + \sqrt{3}$

19 **a**

$f(x)$ is not defined where $\sin x = 0$ or $\cos x = 0$

$f(x)$ is not defined for $x = \frac{n}{2}\pi$ for any $n \in \mathbb{Z}$

b

$$\begin{aligned}\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} &= \frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}\\ &= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}\\ &= 2\end{aligned}$$

Where the function is defined.

20 **a**

$$\begin{aligned}\cos(x + y) + \cos(x - y) &\equiv (\cos x \cos y - \sin x \sin y) + (\cos x \cos y + \sin x \sin y)\\ &\equiv 2\cos x \cos y\end{aligned}$$

b

By part **a**,

$\cos(2x + x) + \cos(2x - x) = 2\cos 2x \cos x$

So $2\cos 2x \cos x = 3\cos 2x$

$\cos 2x\,(2\cos x - 3) = 0$

$\cos 2x = 0$ or $\cos x = \frac{3}{2}$ (reject)

Solutions for $0 \le x \le 2\pi$:

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

21 **a**

Using compound angle formula:

$$\tan(\arctan 3 - \arctan 2) = \frac{3-2}{1+3\times 2} = \frac{1}{7}$$

b

Using double angle formula:

$$\tan\left(2\arctan\left(\frac{1}{2}\right)\right) = \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

22

Angle above the horizontal of the line $y = x$ is $\arctan 1$

Angle above the horizontal of the line $y = 2x$ is $\arctan 2$

The angle between the lines is therefore $\arctan 2 - \arctan 1$

Using compound angle formula for $\tan(A - B)$:

$$\tan(\arctan 2 - \arctan 1) = \frac{2-1}{1+2\times 1} = \frac{1}{3}$$

23 **a**

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ and so $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$$\begin{aligned}\tan 3x &\equiv \tan(x+2x)\\ &\equiv \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}\\ &\equiv \frac{\tan x + \left(\frac{2\tan x}{1-\tan^2 x}\right)}{1 - \tan x\left(\frac{2\tan x}{1-\tan^2 x}\right)}\\ &\equiv \frac{\left(\frac{3\tan x - \tan^3 x}{1-\tan^2 x}\right)}{\left(\frac{1-3\tan^2 x}{1-\tan^2 x}\right)}\\ &\equiv \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}\end{aligned}$$

b

$$\tan x + \frac{3\tan x - \tan^3 x}{1-3\tan^2 x} = 0$$

$(\tan x - 3\tan^3 x) + (3\tan x - \tan^3 x) = 0$

$4\tan x\,(1-\tan^2 x) = 0$

$\tan x = 0$ or $\tan x = \pm 1$

Solutions for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$:

$x = 0$ or $\pm\dfrac{\pi}{4}$

24 **a**

Using compound angle formula:

$$\begin{aligned}\sin\left(2x+\frac{\pi}{2}\right) &= \sin 2x\cos\frac{\pi}{2} + \cos 2x \sin\frac{\pi}{2}\\ &= 0\times\sin 2x + 1\times\cos 2x\\ &= \cos 2x\end{aligned}$$

b

$\sin\left(2x+\frac{\pi}{2}\right)=\sin 3x$

$2x+\frac{\pi}{2}=3x+2n\pi$ or $2x+\frac{\pi}{2}=\pi-3x+2n\pi$

$x=\frac{\pi}{2}-2n\pi$ or $5x=\frac{\pi}{2}+2n\pi$

$x=\frac{\pi}{2}-2n\pi$ or $x=\frac{\pi}{10}+\frac{2n\pi}{5}$

Solutions for $0\leq x\leq\frac{\pi}{2}$:

$x=\frac{\pi}{2}$ or $\frac{\pi}{10}$

c

Using double angle formulae $\sin 2A\equiv 2\sin A\cos A$ and $\cos 2A=\cos^2 A-\sin^2 A$ and compound angle formula $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$

$$\begin{aligned}\sin 3x &\equiv \sin(x+2x)\\ &\equiv \sin x\cos 2x+\cos x\sin 2x\\ &\equiv \sin x\,(\cos^2 x-\sin^2 x)+\cos x\,(2\sin x\cos x)\\ &\equiv 3\sin x\cos^2 x-\sin^3 x\\ &\equiv 3\sin x\,(1-\sin^2 x)-\sin^3 x\\ &\equiv 3\sin x-4\sin^3 x\end{aligned}$$

So

$$\cos 2x-\sin 3x\equiv 1-2\sin^2 x-3\sin x+4\sin^3 x$$

d

$\mathrm{f}(1)=4-2-3+1=0$ so $(s-1)$ is a factor of $\mathrm{f}(s)$ (by the factor theorem)

$$\begin{aligned}\mathrm{f}(s)&=(s-1)(4s^2+2s-1)\\ &=(s-1)\left(2s+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(2s+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\\ &=4(s-1)\left(s-\left(\frac{\sqrt{5}-1}{4}\right)\right)\left(s-\left(-\frac{1+\sqrt{5}}{4}\right)\right)\end{aligned}$$

e

By part **c**,

$\mathrm{f}(\sin x)=\cos 2x-\sin 3x$

But by part **b**, $\mathrm{f}(\sin x)=0$ when $x=\frac{\pi}{10},\frac{\pi}{2}$ so these solutions must coincide with the roots of the cubic.

$\sin\frac{\pi}{2}=1$ corresponds to the factor $(s-1)$

$\sin\frac{\pi}{10}>0$ so this must correspond to the factor $\left(s-\left(\frac{\sqrt{5}-1}{4}\right)\right)$

So $\sin\frac{\pi}{10}=\frac{\sqrt{5}-1}{4}$

25 **a**

For even n, the furthest distance will lie between opposite vertices, which will equal the diameter of the circle x.

The area of the shape is nT where T is the area of the isosceles triangle with equal sides $\frac{x}{2}$ and enclosed angle $\frac{2\pi}{n}$

$T = \frac{1}{2}\left(\frac{x}{2}\right)^2 \sin\left(\frac{2\pi}{n}\right) = \frac{x^2}{8}\sin\left(\frac{2\pi}{n}\right)$

Then $A = \frac{nx^2}{8}\sin\left(\frac{2\pi}{n}\right)$

$C = \frac{4A}{\pi x^2} = \frac{n}{2\pi}\sin\left(\frac{2\pi}{n}\right)$

b

Require $C > 0.99$

If n is even then $\frac{n}{2\pi}\sin\left(\frac{2\pi}{n}\right) > 0.99$, which from GDC has solution $n > 25.6$

So the least even number is $n = 26$

If n is odd then $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$, which from GDC has solution $n > 20.3$

So the least even number is $n = 21$

c

For both even and odd numbers of sides, compactness increases towards 1 as n increases, so this aspect of the compactness definition aligns with expectation. However, the differences between the odd and even values of n illustrate that this measure of compactness is not a good one, as expectation would be for compactness to increase as n increases but $C(21) > C(26)$.

26 **a**

$\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right), p \in \mathbb{Z}^+$

Taking tan of both sides:

$\tan\left(\arctan\frac{1}{5} + \arctan\frac{1}{8}\right) = \tan\left(arctan\frac{1}{p}\right)$

Using compound angle formula for $\tan(A+B)$ and $\tan(\arctan x) = x$:

$\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}} = \frac{1}{p}$

$\frac{1}{p} = \frac{\left(\frac{13}{40}\right)}{\left(\frac{39}{40}\right)} = \frac{1}{3}$

$p = 3$

b

$\tan\left(\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}\right)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\frac{\left(\frac{5}{6}\right)}{\left(\frac{5}{6}\right)}=1$

Since $0<\arctan\frac{1}{2},\arctan\frac{1}{5},\arctan\frac{1}{8}<\frac{\pi}{4}=\arctan 1$,

$0<\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}<\frac{3\pi}{4}$

The only angle θ in this interval for which $\tan\theta=1$ is $\theta=\frac{\pi}{4}$

So $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\frac{\pi}{4}$

Worked solutions

4 Complex numbers

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 4A

26

Using the quadratic formula:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(5)(5)}}{2(5)}$$
$$= \frac{-6 \pm \sqrt{-64}}{10}$$
$$= \frac{-6 \pm 8\text{i}}{10}$$
$$= -\frac{3}{5} \pm \frac{4}{5}\text{i}$$

27

$$z = 7 + 3\text{i} - \frac{10\text{i}}{2 + \text{i}}$$
$$= 7 + 3\text{i} - \frac{10\text{i}(2 - \text{i})}{(2 + \text{i})(2 - \text{i})}$$
$$= 7 + 3\text{i} - \frac{20\text{i} + 10}{4 + 1}$$
$$= 7 + 3\text{i} - 4\text{i} - 2$$
$$= 5 - \text{i}$$
$$z^* = 5 + \text{i}$$

28

$$4z - 23 = 5\text{i}z + 2\text{i}$$
$$(4 - 5\text{i})z = 23 + 2\text{i}$$
$$(4 + 5\text{i})(4 - 5\text{i})z = (23 + 2\text{i})(4 + 5\text{i})$$
$$(16 + 25)z = 92 - 10 + (8 + 115)\text{i}$$
$$41z = 82 + 123\text{i}$$
$$z = 2 + 3\text{i}$$

29

Let $z = x + \text{i}y$ so $z^* = x - \text{i}y$

$$3\text{i}z - 2z^* = \text{i} - 4$$
$$3\text{i}x - 3y - 2x + 2\text{i}y = \text{i} - 4$$

Equating real and complex coefficients:

$$\begin{cases} -3y - 2x = -4 (1) \\ 3x + 2y = 1 (2) \end{cases}$$
$$2(1) + 3(2): 5x = -5$$
$$x = -1, y = 2$$
$$z = -1 + 2\text{i}$$

30

$$z=\frac{a+3\text{i}}{a-3\text{i}}$$
$$=\frac{(a+3\text{i})^2}{(a-3\text{i})(a+3\text{i})}$$
$$=\frac{a^2-9+6a\text{i}}{a^2+9}$$
$$\text{Re}(z)=\frac{a^2-9}{a^2+9}$$

If $\text{Re}(z)=0$ then $a=\pm 3$

31

Let $z=x+\text{i}y$ so $z^*=x-\text{i}y$

$(z^*)^2=(x-\text{i}y)^2$

$=x^2-y^2-2\text{i}xy$

$(z^2)^*=((x+\text{i}y)^2)^*$

$=(x^2-y^2+2\text{i}xy)^*$

$=x^2-y^2-2\text{i}xy$

So $(z^*)^2\equiv(z^2)^*$

32

$$\begin{cases}3z+\text{i}w=5-11\text{i}(1)\\2\text{i}z-3w=-2+\text{i}(2)\end{cases}$$

$2\text{i}(1)-3(2):-2\text{w}+9\text{w}=22+10\text{i}+6-3\text{i}$

$7w=28+7\text{i}$

$w=4+\text{i}$

$(1):3z=5-11\text{i}-\text{i}w=5-11\text{i}+1-4\text{i}=6-15\text{i}$

$z=2-5\text{i}$

33

$$\begin{cases}2z-3\text{i}w=9+\text{i}(1)\\(1+\text{i})z+4w=1+10\text{i}(2)\end{cases}$$

$4(1)+3\text{i}(2):8z+(3\text{i}-3)\text{z}=36+4\text{i}+3\text{i}-30$

$(5+3\text{i})z=6+7\text{i}$

$(5-3\text{i})(5+3\text{i})z=(5-3\text{i})(6+7\text{i})$

$(25+9)z=30+21+(35-18)\text{i}$

$34z=51+17\text{i}$

$$z=\frac{3}{2}+\frac{1}{2}\text{i}$$

$$(2):4w=1+10\text{i}-(1+\text{i})\left(\frac{3}{2}+\frac{1}{2}\text{i}\right)$$
$$=1+10\text{i}-\left(\frac{3}{2}-\frac{1}{2}+\left(\frac{3}{2}+\frac{1}{2}\right)\text{i}\right)$$
$$=8\text{i}$$

$w=2\text{i}$

34

$(1 + ai)(1 + bi) = b - a + 9i$

$1 - ab + (a + b)i = b - a + 9i$

Comparing real and imaginary parts:

$\begin{cases} 1 - ab = b - a (1) \\ a + b = 9 (2) \end{cases}$

$(2): a = 9 - b$

Substituting into (1): $1 - (9 - b)b = b - (9 - b)$

$b^2 - 11b + 10 = 0$

$(b - 1)(b - 10) = 0$

$b = 1$ or 10

Solutions: $a = 8, b = 1$ or $a = -1, b = 10$

35

$$z = \frac{7 + i}{2 - i} - \frac{3 + i}{a + 2i}$$

$$= \frac{(7 + i)(2 + i)}{(2 - i)(2 + i)} - \frac{(3 + i)(a - 2i)}{(a + 2i)(a - 2i)}$$

$$= \frac{13 + 9i}{5} - \frac{3a + 2 + (a - 6)i}{a^2 + 4}$$

$$= \frac{13(a^2 + 4) - 15a - 10}{5(a^2 + 4)} + \frac{9(a^2 + 4) - 5a + 30}{5(a^2 + 4)}i$$

If $\text{Re}(z) = \text{Im}(z)$ then

$13(a^2 + 4) - 15a - 10 = 9(a^2 + 4) - 5a + 30$

$4a^2 - 10a - 24 = 0$

$2a^2 - 5a - 12 = 0$

$(a - 4)(2a + 3) = 0$

$a = 4$ or $-\frac{3}{2}$

36

Let $z = x + iy$

$z^2 = -3 - 4i$

$x^2 - y^2 + 2ixy = -3 - 4i$

Comparing real and imaginary parts:

$\begin{cases} x^2 - y^2 = -3 (1) \\ 2xy = -4 (2) \end{cases}$

$(2): x = -2y^{-1}$

Substituting into (1): $4y^{-2} - y^2 = -3$

$y^4 - 3y^2 - 4 = 0$

$(y^2 - 4)(y^2 + 1) = 0$

$y^2 = 4$ or -1 (reject)

$y = \pm 2$

Then solutions are $z = \pm(1 - 2i)$

37

Let $z = x + iy$

$z^2 = 8 - 6i$

$x^2 - y^2 + 2ixy = 8 - 6i$

Comparing real and imaginary parts:

$\begin{cases} x^2 - y^2 = 8 (1) \\ 2xy = -6 (2) \end{cases}$

$(2): x = -3y^{-1}$

Substituting into (1): $9y^{-2} - y^2 = 8$

$y^4 + 8y^2 - 9 = 0$

$(y^2 - 1)(y^2 + 9) = 0$

$y^2 = 1$ or -9 (reject)

$y = \pm 1$

Then solutions are $z = \pm(3 - i)$

38

Let $z = x + iy$

$z^2 = x^2 - y^2 + 2ixy$ and $z^* = x - iy$

Comparing real and imaginary coefficients when $z^2 = z^*$:

$\begin{cases} x^2 - y^2 = x (1) \\ 2xy = -y (2) \end{cases}$

$(2): x = -\frac{1}{2}$ or $y = 0$

If $y = 0$ then $(1): x^2 = x$ so $x = 1$ or 0

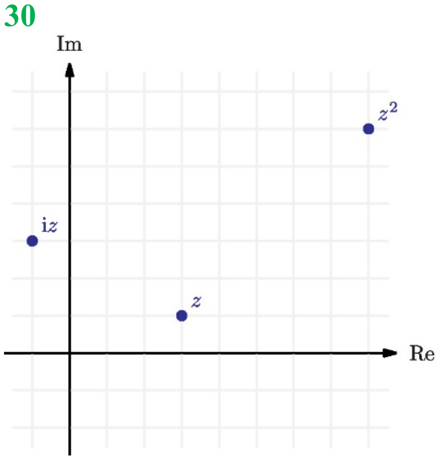
If $x = -\frac{1}{2}$ then $(1): \frac{1}{4} - y^2 = -\frac{1}{2}$ so $y^2 = \frac{3}{4}; y = \pm\frac{\sqrt{3}}{2}$

Solutions:

$z = 0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Exercise 4B

30



31 **a** $|-2 + 2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$

b $\text{Im } z > 0$ so $\arg z = \arctan\left(\frac{2}{-2}\right) = \frac{3\pi}{4}$

c $|z^2| = |z|^2 = 8$

$$\arg z^2 = 2\arg z = \frac{3\pi}{2}$$

d $z^2 = 8\left(\cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right)\right) = -8\mathrm{i}$

32 $\operatorname{cis} 0.6 \times \operatorname{cis} 0.4 = \operatorname{cis} 1$

33 **a** $\operatorname{Im}(w) > 0$ so $\arg w = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

b $\operatorname{Im}(z) > 0$ so $\arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$$\operatorname{Im}(zw) = \operatorname{Im}(z) + \operatorname{Im}(w) = \frac{7\pi}{12}$$

34 **a** $\operatorname{cis}\frac{\pi}{3} \times \operatorname{cis}\frac{\pi}{6} = \operatorname{cis}\frac{\pi}{2} = \mathrm{i}$

b $\operatorname{cis}\frac{\pi}{3} + \operatorname{cis}\frac{\pi}{6} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = \frac{1+\sqrt{3}}{2}(1+\mathrm{i})$

35

$$\operatorname{cis}\left(\frac{\pi}{3}\right) \times \operatorname{cis}\left(\frac{3\pi}{4}\right) = \operatorname{cis}\left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$$
$$= \operatorname{cis}\left(\frac{13\pi}{12}\right)$$
$$= \cos\left(\frac{13\pi}{12}\right) + \mathrm{i}\sin\left(\frac{13\pi}{12}\right)$$

36

$$\frac{\operatorname{cis}\left(\frac{3\pi}{5}\right)}{\operatorname{cis}\left(\frac{\pi}{4}\right)} = \operatorname{cis}\left(\frac{3\pi}{5} - \frac{\pi}{4}\right)$$
$$= \operatorname{cis}\left(\frac{7\pi}{20}\right)$$
$$= \cos\left(\frac{7\pi}{20}\right) + \mathrm{i}\sin\left(\frac{7\pi}{20}\right)$$

37 Let $z = r\operatorname{cis}\theta$

Then $\arg z = \theta$ and $\arg z^* = 2\pi - \theta$, if $\arg z$ is defined to have range $[0, 2\pi)$

So $\arg z + \arg z^* = 2\pi$

38

If Re (z) > 0 then $\arg z = \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$

(adjusting by adding 2π if necessary to ensure the argument lies in the appropriate interval)

If Re (z) < 0 then $\arg z = \pi + \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$

So when $a, b < 0$: $\arg(a + \mathrm{i}b) = \pi + \arctan\left(\frac{b}{a}\right)$

39

$$\mathrm{i}\operatorname{cis}\theta = \operatorname{cis}\left(\frac{\pi}{2}\right)\operatorname{cis}\theta$$
$$= \operatorname{cis}\left(\theta + \frac{\pi}{2}\right)$$

40

$$1 + \mathrm{i}\tan\theta = \frac{\cos\theta + \mathrm{i}\sin\theta}{\cos\theta}$$
$$= \sec\theta \operatorname{cis}\theta$$

41

For example, $z_1 = z_2 = \mathrm{i}$

Then $\arg z_1 = \arg z_2 = \arg(z_1 + z_2) = \frac{\pi}{2}$

So $\arg z_1 + \arg z_2 = \pi \neq \arg(z_1 + z_2)$

42

$$|zw| = |z||w| = \sqrt{\left(-4\sqrt{2}\right)^2 + \left(4\sqrt{2}\right)^2} = 8$$

$$|zw^{-1}| = |z||w|^{-1}\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

Then $\left|\frac{zw}{zw^{-1}}\right| = 4 = |w|^2$ so $|w| = 2$ and $|z| = 4$

$$\arg(zw) = \arg z + \arg w = \frac{\pi}{2} + \arctan\left(\frac{4\sqrt{2}}{-4\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$\arg(zw^{-1}) = \arg z - \arg w = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

Then $\arg\left(\frac{zw}{zw^{-1}}\right) = 2\arg w = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$ so $\arg w = \frac{5\pi}{24}$ and $\arg z = \frac{13\pi}{24}$

43

Tip: Two methods are offered; the first is brute-force and long.

The second uses a more intelligent argument to reduce the amount of algebra required.

Always look for a fast solution, but be sure to make your explanation clear!

<u>Method 1: Algebraic</u>

$|z| = \sqrt{6^2 + 8^2} = 10$

Let $w = u + \mathrm{i}v$, so that $u^2 + v^2 = 25$, from which $u^2 = 25 - v^2$ $(*)$

$|z + w| = \sqrt{(6+u)^2 + (8+v)^2} = 15$

$(6+u)^2 + (8+v)^2 = 225$

$36 + 12u + u^2 + 64 + 16v + v^2 = 225$

$12u + 16v + u^2 + v^2 = 125$

$12u + 16v = 100$

$3u = 25 - 4v$

Squaring and then substituting $(*)$:

$9u^2 = (25 - 4v)^2 = 9(25 - v^2)$

$625 - 200v + 16v^2 = 225 - 9v^2$

$25v^2 - 200v + 400 = 0$

$v^2 - 8v + 16 = 0$

$(v - 4)^2 = 0$

$v = 4$ so $u = 3$

$w = 3 + 4\mathrm{i}$

Method 2:

For any two complex numbers, $|z+w| \le |z| + |w|$ by the triangle inequality.

Equality occurs only when $0, z$ and w are collinear, that is, when $w = kz$ for some non-negative real value k.

$|z| = \sqrt{6^2 + 8^2} = 10$

Since $|w| = \frac{1}{2}|z|$, it follows that if $w = kz$ for some $k \in \mathbb{R}^+$, then $w = \frac{1}{2}z$

Hence $w = 3 + 4\mathrm{i}$

44 **a**

$$\frac{1}{\cos x + \mathrm{i}\sin x} = \frac{(\cos x - \mathrm{i}\sin x)}{(\cos x - \mathrm{i}\sin x)} \times \frac{1}{(\cos x + \mathrm{i}\sin x)}$$
$$= \frac{(\cos x - \mathrm{i}\sin x)}{\cos^2 x + \sin^2 x}$$
$$= \cos x - \mathrm{i}\sin x$$

b

$|z| = 1$

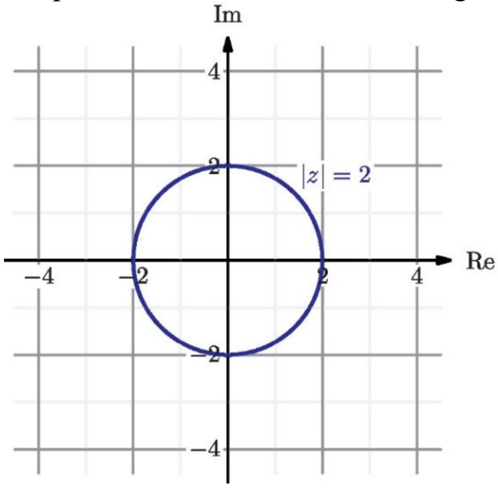
From part **a**,

$$\frac{1}{z} = \mathrm{Re}\,(z) - \mathrm{Im}\,(z)$$

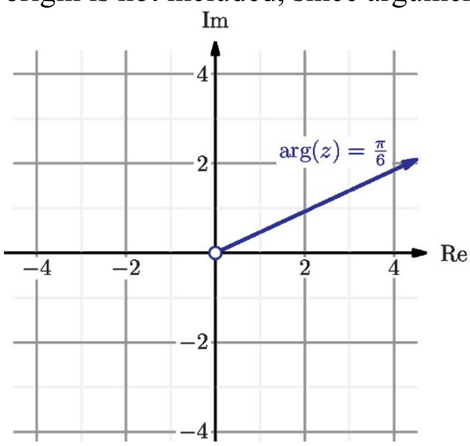
$$|z| + \frac{1}{|z|} = 2\,\mathrm{Re}\,(z)$$

45 **a**

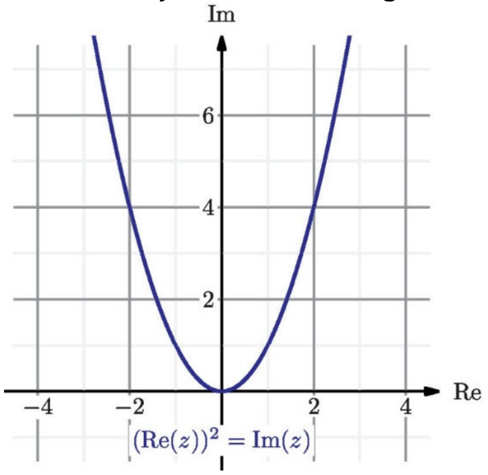
All points a distance 2 from the origin:



b All points on the half-line at angle 30° from the origin. Note that the origin is not included, since argument is not defined for $z = 0$.



c Since Re (z) is given on the x axis and Im (z) is given on the y axis, this is the curve $y = x^2$ on the Argand diagram.



46

$|z - 5\text{i}| = 3$

The distance of z from the point 5i on the Argand diagram is 3.

The point on this circle closest to the origin (with least modulus value) is $z = 2\text{i}$ so $|z| = 2$.

47

$$\frac{1+\text{e}^{2\text{i}x}}{1-\text{e}^{2\text{i}x}} = \frac{1+\cos 2x + \text{i}\sin 2x}{1-\cos 2x - \text{i}\sin 2x}$$
$$= \frac{1+\cos 2x + \text{i}\sin 2x}{1-\cos 2x - \text{i}\sin 2x} \times \frac{1-\cos 2x + \text{i}\sin 2x}{1-\cos 2x + \text{i}\sin 2x}$$
$$= \frac{(1+\text{i}\sin 2x)^2 - (\cos 2x)^2}{(1-\cos 2x)^2 + (\sin 2x)^2}$$
$$= \frac{1-\sin^2 2x + 2\text{i}\sin 2x - \cos^2 2x}{1+\cos^2 2x - 2\cos 2x + \sin^2 2x}$$
$$= \frac{2\text{i}\sin 2x}{2-2\cos 2x}$$
$$= \frac{4\text{i}\sin x\cos x}{4\sin^2 x}$$
$$= \frac{\text{i}\cos x}{\sin x}$$
$$= \text{i}\cot x$$

48 **ai**

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$$

aii

$$\text{Re}\, z > 0 \text{ so } \arg z = \arctan\left(\frac{\text{Im}\, z}{\text{Re}\, z}\right) = \arctan 1 = \frac{\pi}{4}$$

aiii

$$|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = 2$$

aiv

$\text{Re}\, w > 0$ so $\arg w = \arctan\left(\frac{\text{Im}\, w}{\text{Re}\, w}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$

b

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = 2$$

$$\arg\left(\frac{w}{z}\right) = \arg w - \arg z = \frac{\pi}{12}$$

So $\frac{w}{z} = 2\,\text{cis}\,\frac{\pi}{12}$

c

$$\frac{w}{z} = \frac{1 + \mathrm{i}\sqrt{3}}{\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(\frac{1 + \mathrm{i}\sqrt{3}}{1 + \mathrm{i}}\right)$$

$$= \sqrt{2}\left(\frac{1 + \mathrm{i}\sqrt{3}}{1 + \mathrm{i}} \times \frac{1 - \mathrm{i}}{1 - \mathrm{i}}\right)$$

$$= \frac{\sqrt{2}\left(1 + \sqrt{3} + \mathrm{i}(\sqrt{3} - 1)\right)}{2}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{2}} + \mathrm{i}\frac{\sqrt{3} - 1}{\sqrt{2}}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{2} + \mathrm{i}\frac{\sqrt{6} - \sqrt{2}}{2}$$

d

From part **b**, $\frac{w}{z} = 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

Equating the real part with the result in part **c**,

$$\cos\frac{\pi}{12} = \frac{1}{2}\left(\frac{\sqrt{2} + \sqrt{6}}{2}\right) = \frac{\sqrt{2} + \sqrt{6}}{4}$$

49

Let $z = x + \mathrm{i}y$

$\sqrt{x^2 + y^2} + x + \mathrm{i}y = 8 + 4\mathrm{i}$

Comparing real and imaginary parts:

$$\begin{cases} \sqrt{x^2 + y^2} + x = 8 \quad (1) \\ y = 4 \quad (2) \end{cases}$$

(1): $\sqrt{16 + x^2} = 8 - x$

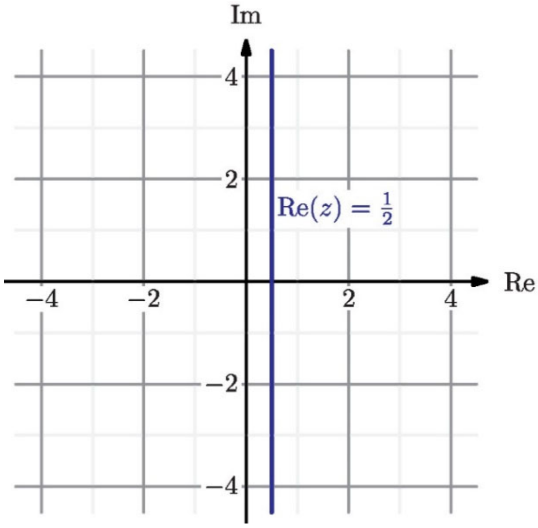
Squaring: $16 + x^2 = 64 - 16x + x^2$

$16x = 48$

$x = 3$

$z = 3 + 4\mathrm{i}$

50 **a**
Let $z = x + \mathrm{i}y$
$\sqrt{x^2 + y^2} = \sqrt{(x-1)^2 + y^2}$
$x^2 + y^2 = x^2 - 2x + 1 + y^2$
$x = \mathrm{Re}(z) = \dfrac{1}{2}$
b



51 **a** Each other point is found by rotating w repeatedly through $120° = \frac{2\pi}{3}$ radians.

If one vertex is $z_1 = w$ then the other two vertices are $z_2 = we^{\frac{2i\pi}{3}}$ and $z_3 = we^{\frac{4i\pi}{3}}$

b The length of a side equals

$$|z_1 - z_2| = \left|w\left(1 - e^{\frac{2i\pi}{3}}\right)\right|$$
$$= |w|\left|1 - \cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right|$$
$$= |w|\left|\frac{3}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right|$$
$$= |w|\sqrt{\frac{9}{4} + \frac{3}{4}}$$
$$= |w|\sqrt{3}$$

52
$$3^{\mathrm{i}} = \left(e^{\ln 3}\right)^{\mathrm{i}}$$
$$= e^{\mathrm{i}\ln 3}$$
$$= 1\,\mathrm{cis}(\ln 3)$$

53 **a** $\mathrm{i} = e^{\frac{\mathrm{i}\pi}{2}}$ so $\mathrm{i}^{\mathrm{i}} = e^{-\frac{\pi}{2}} \in \mathbb{R}$

b Using the approximation $e \approx \pi \approx 3$, $\mathrm{i}^{\mathrm{i}} \approx 3^{-1.5} \approx 0.2$

54 **a** $-2 = 2e^{\mathrm{i}\pi}$

b $\ln(-2) = \ln\left(2e^{\mathrm{i}\pi}\right) = \ln 2 + \mathrm{i}\pi$

55 **a** $\mathrm{i} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}$

b $\ln \mathrm{i} = \frac{\mathrm{i}\pi}{2}$

c The argument is not unique; more generally, $\mathrm{i} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)}$ for $n \in \mathbb{Z}$

So $\ln \mathrm{i} = \mathrm{i}\left(\frac{\pi}{2} + 2n\pi\right)$ for any $n \in \mathbb{Z}$.

56 **a** $\mathrm{Re}\left(\mathrm{e}^{(1+\mathrm{i})x}\right) = \mathrm{Re}\,(\mathrm{e}^x \operatorname{cis} x)$

$$= \mathrm{e}^x \cos x$$

b

$$\begin{aligned}\int \mathrm{e}^x \sin x \;\mathrm{d}x &= \int \mathrm{Im}\left(\mathrm{e}^{(1+\mathrm{i})x}\right) \mathrm{d}x \\ &= \mathrm{Im} \int \mathrm{e}^{(1+\mathrm{i})x} \;\mathrm{d}x \\ &= \mathrm{Im}\left(\frac{1}{1+\mathrm{i}}\mathrm{e}^{(1+\mathrm{i})x}\right) \\ &= \mathrm{Im}\left(\frac{1-\mathrm{i}}{2}\mathrm{e}^x \operatorname{cis} x\right) \\ &= \frac{\mathrm{e}^x}{2}(\sin x - \cos x)\end{aligned}$$

In the above calculation, we are treating i as a constant value and integrating the function accordingly. This may seem naïve and inappropriate, but in fact the approach is valid in this sort of simple example, and can offer a tidy and rapid method for integrating otherwise awkward functions.

You could achieve the same result here, albeit at greater length, using integration by parts twice; see Chapter 10 for this method.

57 **a**

$$C = \sum_{k=0}^{k=n} \cos k\theta \;\text{ and } S = \sum_{k=0}^{k=n} \sin k\theta$$

$$\cos k\theta + \mathrm{i} \sin k\theta = \mathrm{e}^{ik\theta}$$

$$\text{So } C + \mathrm{i}S = \sum_{k=0}^{k=n} \mathrm{e}^{\mathrm{i}k\theta}$$

This is a geometric series with common ratio $\mathrm{e}^{\mathrm{i}\theta}$ and first term 1.

b
Using the formula for the sum of a geometric series, noting that the sum given contains $n+1$ terms:

$$C + \mathrm{i}S = \frac{1 - \mathrm{e}^{\mathrm{i}(n+1)\theta}}{1 - \mathrm{e}^{\mathrm{i}\theta}}$$
$$= \frac{1 - \cos\big((n+1)\theta\big) - \mathrm{i}\sin\big((n+1)\theta\big)}{1 - \cos\theta - \mathrm{i}\sin\theta}$$
$$= \frac{1 - \cos\big((n+1)\theta\big) - \mathrm{i}\sin\big((n+1)\theta\big)}{1 - \cos\theta - \mathrm{i}\sin\theta} \times \frac{1 - \cos\theta + \mathrm{i}\sin\theta}{1 - \cos\theta + \mathrm{i}\sin\theta}$$
$$= \frac{\Big(1 - \cos\big((n+1)\theta\big) - \mathrm{i}\sin\big((n+1)\theta\big)\Big)(1 - \cos\theta + \mathrm{i}\sin\theta)}{(1-\cos\theta)^2 + \sin^2\theta}$$
$$= \frac{\Big(1 - \cos\big((n+1)\theta\big) - \mathrm{i}\sin\big((n+1)\theta\big)\Big)(1 - \cos\theta + \mathrm{i}\sin\theta)}{1 + \cos^2\theta + \sin^2\theta - 2\cos\theta}$$

$C = \mathrm{Re}\,(C + \mathrm{i}S)$
$$= \frac{\Big(1 - \cos\big((n+1)\theta\big)\Big)(1 - \cos\theta) + \sin\big((n+1)\theta\big)\sin\theta}{2 - 2\cos\theta}$$
$$= \frac{1 - \cos\theta - \cos\big((n+1)\theta\big) + \Big[\cos\theta\cos\big((n+1)\theta\big) + \sin\big((n+1)\theta\big)\sin\theta\Big]}{2 - 2\cos\theta}$$
$$= \frac{1 - \cos\theta - \cos(n+1)\theta + \cos\big((n+1)\theta - \theta\big)}{2 - 2\cos\theta}$$
$$= \frac{(1 - \cos\theta + \cos n\theta - \cos((n+1)\,\theta)}{2 - 2\cos\theta}$$

58 **a** $z = \mathrm{i} + \frac{4}{w}$

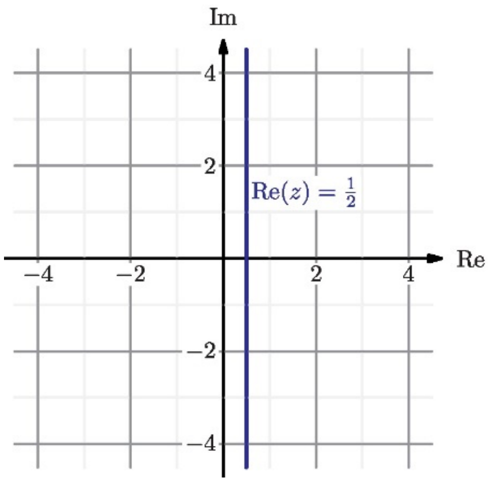
b
$z = x + \mathrm{i}y$ where $x^2 + y^2 = 1$

$$w = \frac{4}{z - \mathrm{i}} = \frac{4\big(x + \mathrm{i}(1-y)\big)}{\big(x + \mathrm{i}(y-1)\big)\big(x + \mathrm{i}(1-y)\big)}$$

$$\mathrm{Im}(w) = \frac{4(1-y)}{x^2 + (y-1)^2} = \frac{4(1-y)}{x^2 + y^2 - 2y + 1} = \frac{4(1-y)}{2 - 2y} = 2$$

(Solution fails if $y = 1, x = 0$ so that $z = \mathrm{i}$ but in that case w is not defined)

59



Exercise 4C

13 **a**

If $x = 2$ is a root then $(x - 2)$ is a factor of $\mathrm{p}(x)$

$\mathrm{p}(x) = (x - 2)(ax^2 + bx + c) = x^3 - 8x^2 + 22x - 20$

Comparing coefficients:

$x^3: a = 1$

$x^2: b - 2a = -8$ so $b = 2a - 8 = -6$

$x^1: c - 2b = 22$ so $c = 2b + 22 = 10$

$x^0: -2c = -20$ is consistent

$\mathrm{p}(x) = (x - 2)(x^2 - 6x + 10)$

b

Using the quadratic formula:

$x = 2$ or $3 \pm \mathrm{i}$

14 **a**

If $x = -2$ is a root then $(x + 2)$ is a factor of $\mathrm{p}(x)$

$\mathrm{p}(x) = (x + 2)(ax^2 + bx + c) = x^3 - 8x^2 + 9x + 58$

Comparing coefficients:

$x^3: a = 1$

$x^2: b + 2a = -8$ so $b = -8 - 2a = -10$

$x^1: c + 2b = 9$ so $c = 9 - 2b = 29$

$x^0: 2c = 58$ is consistent

$\mathrm{p}(x) = (x + 2)(x^2 - 10x + 29)$

b

Using the quadratic formula:

$x = -2$ or $5 \pm 2\mathrm{i}$

15 **a**

$\mathrm{p}(x) = (2x - 1)(ax^2 + bx + c) = 2x^3 + 7x^2 + 8x - 6$

Comparing coefficients:

$x^3: 2a = 2$ so $a = 1$

$x^2: 2b - a = 7$ so $b = 0.5(7 + a) = 4$

$x^1: 2c - b = 8$ so $c = 0.5(8 + b) = 6$

$x^0: -c = -6$ is consistent

$\mathrm{p}(x) = (2x - 1)(x^2 + 4x + 6)$

b

$\mathrm{p}(x) = (2x - 1)\left(x + 2 + \mathrm{i}\sqrt{2}\right)\left(x + 2 - \mathrm{i}\sqrt{2}\right)$

So the solutions to $\mathrm{p}(x) = 0$ are $x = \frac{1}{2}, -2 \pm \mathrm{i}\sqrt{2}$

16 **a**

For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $1 - 4\mathrm{i}$ is a root, $1 + 4\mathrm{i}$ is also a root.

b

$$\begin{aligned} x^3 + x^2 + 11x + 51 &= a(x - 1 + 4\mathrm{i})(x - 1 - 4\mathrm{i})(x - z) \\ &= a(x^2 - 2x + 17)(x - z) \end{aligned}$$

Comparing coefficients:

$x^3: 1 = a$

$x^2: 1 = a(-z - 2)$ so $z = -3$

$x^1: 11 = a(17 + 2z)$ is consistent with $z = -3$

$x^0: 51 = -17az$ is consistent with $z = -3$

The third linear factor is $(x + 3)$ so the third root is $x = -3$.

17 **a** $\text{p}(3\text{i}) = -27\text{i} - 36 + 27\text{i} + 36 = 0$

b For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given 3i is a root, -3i is also a root.

$$\text{p}(x) = x^3 + 4x^2 + 9x + 36 = a(x - 3\text{i})(x + 3\text{i})(x - z)$$
$$= a(x^2 + 9)(x - z)$$

Comparing coefficients:

$x^3: 1 = a$

$x^2: 4 = -az$ so $z = -4$

$x^1: 9 = 9a$ is consistent with $a = 1$

$x^0: 36 = -9az$ is consistent with $z = -4$

The third linear factor is $(x + 4)$ so the third root is $x = -4$.

The three roots are $x = \pm 3\text{i}, -4$

18 **a** $$\text{p}(x) = x^4 + 3x^3 - x^2 - 13x - 10 = (x + 1)(x - 2)(ax^2 + bx + c)$$
$$= (x^2 - x - 2)(ax^2 + bx + c)$$

Comparing coefficients:

$x^4: 1 = a$

$x^3: 3 = b - a$ so $b = 4$

$x^2: -1 = c - b - 2a$ so $c = 5$

$x^1: -13 = -c - 2b$ is consistent with $b = 4, c = 5$

$x^0: -10 = -2c$ is consistent with $c = 5$

$\text{p}(x) = (x + 1)(x - 2)(x^2 + 4x + 5)$

b

$\text{p}(x) = (x + 1)(x - 2)(x + 2 + \text{i})(x + 2 - \text{i})$

The roots are $-1, 2, -2 \pm \text{i}$

19 **a**

$\text{p}(x) = x^4 - 3x^3 + 8x - 24$

-2 and 3 are roots, so $(x + 2)$ and $(x - 3)$ are factors of $\text{p}(x)$

$$\text{p}(x) = x^4 - 3x^3 + 8x - 24 = (x + 2)(x - 3)(ax^2 + bx + c)$$
$$= (x^2 - x - 6)(ax^2 + bx + c)$$

Comparing coefficients:

$x^4: 1 = a$

$x^3: -3 = b - a$ so $b = -2$

$x^2: 0 = c - b - 6a$ so $c = 4$

$x^1: 8 = -c - 6b$ is consistent with $b = -2, c = 4$

$x^0: -24 = -6c$ is consistent with $c = 4$

$$\text{p}(x) = (x + 2)(x - 3)(x^2 - 2x + 4)$$
$$= (x + 2)(x - 3)\left(x - 1 + \text{i}\sqrt{3}\right)\left(x - 1 - \text{i}\sqrt{3}\right)$$

The other two roots are $1 \pm \text{i}\sqrt{3}$

b

$\text{p}(x) = (x + 2)(x - 3)\left(x - 1 + \text{i}\sqrt{3}\right)\left(x - 1 - \text{i}\sqrt{3}\right)$

20 **a** For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $2 + 5\text{i}$ is a root, $2 - 5\text{i}$ is also a root.

b

If $2 + 5\text{i}$ and $2 - 5\text{i}$ are roots then $(x - 2 - 5\text{i})$ and $(x - 2 + 5\text{i})$ are factors of the polynomial

$$x^4 - 4x^3 + 30x^2 - 4x + 29 = (x - 2 - 5\text{i})(x - 2 + 5\text{i})(ax^2 + bx + c)$$
$$= (x^2 - 4x + 29)(ax^2 + bx + c)$$

Comparing coefficients:
x^4: $1 = a$
x^3: $-4 = b - 4a$ so $b = 0$
x^2: $30 = c - 4b + 29a$ so $c = 1$
x^1: $-4 = -4c + 29b$ is consistent with $b = 0, c = 1$
x^0: $29 = 29c$ is consistent with $c = 1$
$\mathrm{p}(x) = (2+5\mathrm{i})(2-5\mathrm{i})(x^2+1)$
$\quad = (2+5\mathrm{i})(2-5\mathrm{i})(x-\mathrm{i})(x+\mathrm{i})$
The remaining two roots are $\pm\mathrm{i}$

21 **a** For a polynomial with real coefficients, complex roots occur in conjugate pairs
So, given 2i is a root, $-2\mathrm{i}$ is also a root; also, if $4-\mathrm{i}$ is a root then so is $4+\mathrm{i}$.
b
$x^4 - 8x^3 + 21x^2 - 32x + 68 = (x-2\mathrm{i})(x+2\mathrm{i})(x-4+\mathrm{i})(x-4-\mathrm{i})$
$\quad = (x^2+4)(x^2-8x+17)$

22
For a polynomial with real coefficients, complex roots occur in conjugate pairs
So, given $4+\mathrm{i}$ is a root, $4-\mathrm{i}$ is also a root.
Polynomial is
$\mathrm{p}(x) = (x-3)(x-4-\mathrm{i})(x-4+\mathrm{i})$
$\quad = (x-3)(x^2-8x+17)$
$\quad = x^3 - 11x^2 + 41x - 51$

23
For a polynomial with real coefficients, complex roots occur in conjugate pairs
So, given $3-3\mathrm{i}$ is a root, $3+3\mathrm{i}$ is also a root.
Polynomial is
$\mathrm{f}(x) = (x+1)(x-3+3\mathrm{i})(x-3-3\mathrm{i})$
$\quad = (x+1)(x^2-6x+18)$
$\quad = x^3 - 5x^2 + 12x + 18$
$b = -5, c = 12, d = 18$

24
For a polynomial with real coefficients, complex roots occur in conjugate pairs
So, given 4i is a root, $-4\mathrm{i}$ is also a root; also, if $2-3\mathrm{i}$ is a root then so is $2+3\mathrm{i}$.
Polynomial is
$\mathrm{p}(x) = a(x-4\mathrm{i})(x+4\mathrm{i})(x-2+3\mathrm{i})(x-2-3\mathrm{i})$
$\quad = a(x^2+16)(x^2-4x+13)$
$\quad = a(x^4 - 4x^3 + 29x^2 - 64x + 208)$
where a is any non-zero real value.

25
For a polynomial with real coefficients, complex roots occur in conjugate pairs
So, given 3i is a root, $-3\mathrm{i}$ is also a root; also, if $2-\mathrm{i}$ is a root then so is $2+\mathrm{i}$.
Polynomial is
$\mathrm{f}(x) = (x-3\mathrm{i})(x+3\mathrm{i})(x-2+\mathrm{i})(x-2-\mathrm{i})$
$\quad = (x^2+9)(x^2-4x+5)$
$\quad = x^4 - 4x^3 + 14x^2 - 36x + 45$
$b = -4, c = 14, d = -36, e = 45$

26

$x^4 + 13x^2 + 40 = 0$
$(x^2 + 5)(x^2 + 8) = 0$
$x^2 = -5 \text{ or } -8$
$x = \pm \mathrm{i}\sqrt{5} \text{ or } \pm \mathrm{i}2\sqrt{2}$

27

A simple example would be $\mathrm{i}x^2 + \mathrm{i} = 0$
The roots are $\pm\mathrm{i}$, but the coefficients are all complex.

An example of a polynomial with some, but not all, complex coefficients where all complex roots have conjugates which are also roots can be made by asymmetrically repeating roots (this breaks the notion of exclusive 'pairs' of complex conjugate roots, but that exclusivity was not mentioned in the question).
So, for example, $(x-\mathrm{i})^2(x+\mathrm{i}) = 0$ has roots $\pm\mathrm{i}$, but has expanded form $x^3 - \mathrm{i}x^2 + x - \mathrm{i} = 0$

Exercise 4D

9 **a**

$|2-2\mathrm{i}| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$\text{Re}\,(2-2\mathrm{i}) > 0 \text{ so } \arg(2-2\mathrm{i}) = \arctan\left(-\frac{2}{2}\right) = -\frac{\pi}{4}$

$z = 2 - 2\mathrm{i} = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

b

$$\begin{aligned} z^5 &= \left(2\sqrt{2}\right)^5 \text{cis}\left(-\frac{5\pi}{4}\right) \\ &= 128\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) \\ &= -128 + 128\mathrm{i} \end{aligned}$$

10 **a**

$\left|\sqrt{3} + \mathrm{i}\right| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\text{Re}\left(\sqrt{3} + \mathrm{i}\right) > 0 \text{ so } \arg\left(\sqrt{3} + \mathrm{i}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$z = \sqrt{3} + \mathrm{i} = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

b

$$\begin{aligned} z^{-3} &= (2)^{-3}\,\text{cis}\left(-\frac{3\pi}{6}\right) \\ &= \frac{1}{8}(-\mathrm{i}) \\ &= -\frac{1}{8}\mathrm{i} \end{aligned}$$

11 **a**

$\left|-\sqrt{2}-\mathrm{i}\sqrt{2}\right| = \sqrt{\left(-\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2} = 2$

$\mathrm{Re}\left(-\sqrt{2}-\mathrm{i}\sqrt{2}\right) < 0$ so $\arg(2-2\mathrm{i}) = \pi + \arctan\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = \frac{5\pi}{4}$

$w = -\sqrt{2}-\mathrm{i}\sqrt{2} = 2\,\mathrm{cis}\left(\frac{5\pi}{4}\right)$

b

$w^6 = 2^6\,\mathrm{cis}\left(\frac{30\pi}{4}\right) = 64\,\mathrm{cis}\left(-\frac{\pi}{2}\right) = -64\mathrm{i}$

$z = \mathrm{cis}\left(\frac{\pi}{7}\right)$ so $z^7 = \mathrm{cis}(\pi) = -1$

Then $w^6z^7 = 64\mathrm{i}$

12 **a**

$z^6 = 1 = 1\,\mathrm{cis}(2n\pi)$ for $n \in \mathbb{Z}$

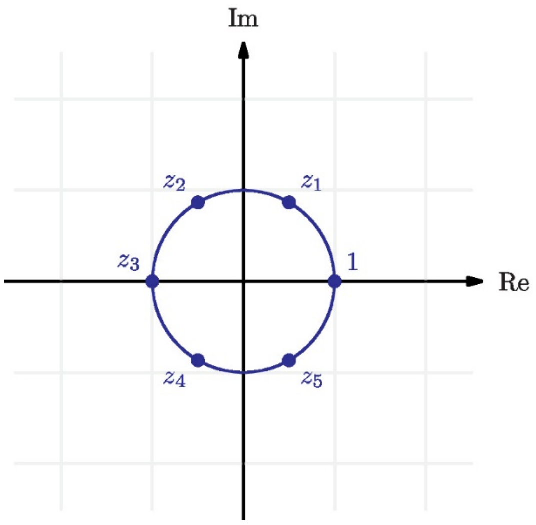
Then $z = (1\,\mathrm{cis}(2n\pi))^{\frac{1}{6}}$

$= 1\,\mathrm{cis}\left(\frac{2n\pi}{6}\right)$

$= \mathrm{cis}\,0, \mathrm{cis}\left(\pm\frac{\pi}{3}\right), \mathrm{cis}\left(\pm\frac{2\pi}{3}\right), \mathrm{cis}(\pi)$

$= \pm 1, \pm\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right)$

b



13 **a**

$z^4 = -16 = 2^4\,\mathrm{cis}(\pi + 2n\pi)$ for $n \in \mathbb{Z}$

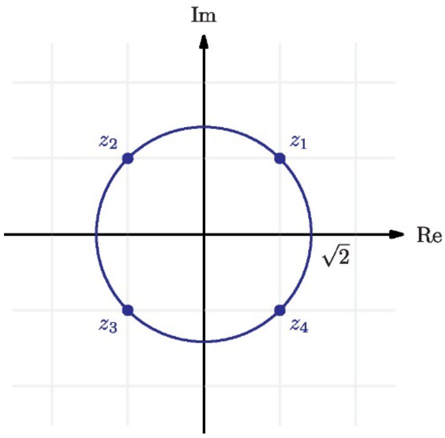
Then $z = (2^4\,\mathrm{cis}(\pi + 2n\pi))^{\frac{1}{4}}$

$= 2\,\mathrm{cis}\left(\frac{\pi + 2n\pi}{4}\right)$

$= 2\,\mathrm{cis}\left(\pm\frac{\pi}{4}\right), 2\,\mathrm{cis}\left(\pm\frac{3\pi}{4}\right)$

$= \sqrt{2} \pm \sqrt{2}i$ or $-\sqrt{2} \pm \sqrt{2}i$

b



14 **a**

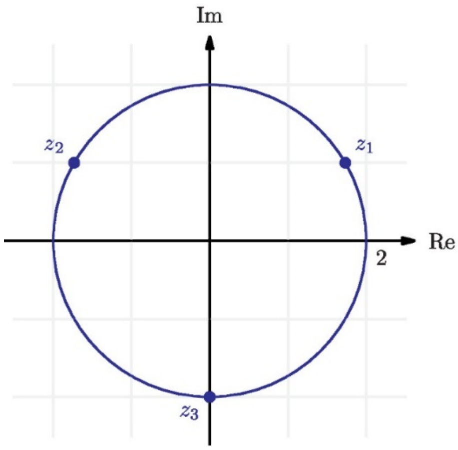
$z^3 = 8\text{i} = 2^3\text{e}^{\left(\frac{\pi}{2}+2n\pi\right)\text{i}}$ for $n \in \mathbb{Z}$

Then $z = \left(2^3\text{e}^{\left(\frac{\pi}{2}+2n\pi\right)\text{i}}\right)^{\frac{1}{3}}$

$= 2\text{e}^{\left(\frac{\pi+4n\pi}{6}\right)\text{i}}$

$= 2\text{e}^{\frac{\pi}{6}\text{i}}, 2\text{e}^{\frac{5\pi}{6}\text{i}}, 2\text{e}^{-\frac{\pi}{2}\text{i}}$

b



15

Five evenly spread vertices must represent fifth roots.

$\left(-\text{i}\sqrt{3}\right)^5 = w = \left(-9\sqrt{3}\right)\text{i}$

So $n = 5, w = \left(-9\sqrt{3}\right)\text{i}$

16

$|1-\mathrm{i}\sqrt{3}| = \sqrt{1^2+(-\sqrt{3})^2} = 2$

$\text{Re}\,(1-\mathrm{i}\sqrt{3}) > 0$ so $\arg(1-\mathrm{i}\sqrt{3}) = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

So $1-\mathrm{i}\sqrt{3} = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$

Then $8(1-\mathrm{i}\sqrt{3})^{-5} = 8\times\left(2^{-5}\,\text{cis}\left(\frac{5\pi}{3}\right)\right)$

$= \frac{1}{4}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$

$= \frac{1}{8}-\frac{\sqrt{3}}{8}\mathrm{i}$

17

$|3-3\mathrm{i}| = \sqrt{3^2+(-3)^2} = 3\sqrt{2}$

$\text{Re}\,(3-3\mathrm{i}) > 0$ so $\arg(3-3\mathrm{i}) = \arctan\left(\frac{-3}{3}\right) = -\frac{\pi}{4}$

So $w = 3-3\mathrm{i} = 3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

Then $w^4 = (3\sqrt{2})^4\,\text{cis}(-\pi) = -324$

$z = \text{cis}\left(\frac{3\pi}{8}\right)$ so $z^6 = \text{cis}\left(\frac{9\pi}{4}\right) = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}$

Hence $w^4z^6 = -162\sqrt{2}-(162\sqrt{2})\mathrm{i}$

18

$z = 2\,\text{cis}\frac{7\pi}{24}$ so $z^n = 2^n\,\text{cis}\frac{7n\pi}{24}$

For this to be a real number, $\frac{7n}{24}\in\mathbb{Z}$

The least value $n\in\mathbb{Z}^+$ under this condition is $n = 24$.

19

$z = \text{cis}\frac{5\pi}{18}$ so $z^n = \text{cis}\frac{5n\pi}{18}$

$\mathrm{i} = \text{cis}\left(\frac{\pi}{2}+2k\pi\right)$ for any $k\in\mathbb{Z}$

Require $\frac{5n}{18} = \frac{1}{2}+2k$

$5n = 9+36k$

The least such $n\in\mathbb{Z}^+$ is $n = 9$, with $k = 1$.

20 **a** $\omega^0 = 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$

b No such integer k exists; the Argand diagram of ω^k cycles through the values given in part **a**.

Alternative reasoning – proof by contradiction: Suppose $k\in\mathbb{Z}$ such that $\omega^k = -1$.

Then $\omega^{2k} = 1$.

$\omega^{2k} = \omega^{7m}$ for some integer m

$2k = 7m(*)$

Both sides of $(*)$ are integer values. Since 2 is not a factor of (prime number) 7, it follows that 7 must be a factor of k and m must be even. But then $k = 7n$ for some integer n.

So $\omega^k = \omega^{7n} = (1)^n = 1$ which contradicts the assumption.
Conclusion: There is no integer k for which $\omega^k = -1$.

c $\omega^{24} = \omega^{21} \times \omega^3 = (\omega^7)^3 \times \omega^3 = 1^3 \times \omega^3 = \omega^3$

$p = 3$

d $(\omega^2)^* = \omega^5$ from the Argand diagram of powers of ω.

$m = 5$.

21 **a** $\omega^5 = 1$

$$(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4)$$
$$= 1+\omega+\omega^2+\omega^3+\omega^4-(\omega+\omega^2+\omega^3+\omega^4+\omega^5)$$
$$= 1+\omega+\omega^2+\omega^3+\omega^4$$
$$-(\omega+\omega^2+\omega^3+\omega^4+1)$$
$$= 0$$

Since $\omega \neq 1$, it follows that $1+\omega+\omega^2+\omega^3+\omega^4 = 0$

b

Let $\omega = \text{cis}\left(\frac{2\pi}{5}\right)$

Then $\omega^k = \text{cis}\left(\frac{2k\pi}{5}\right)$

$$\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\cos\left(\frac{6\pi}{5}\right)+\cos\left(\frac{8\pi}{5}\right) = \text{Re}\,(\omega+\omega^2+\omega^3+\omega^4)$$
$$= \text{Re}\,(-1) \text{ (by part } \mathbf{a}\text{)}$$
$$= -1$$

22 **a**

$z^3 = 1 = \text{cis}(2n\pi)$ for $n \in \mathbb{Z}$

$z = \text{cis}\left(\frac{2n\pi}{3}\right) = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}$

b

$(z-1)^3 = (z+2)^3$

$z - 1 = (z+2)\,\text{cis}\left(\frac{2n\pi}{3}\right)$

$z\left(1 - \text{cis}\left(\frac{2n\pi}{3}\right)\right) = 1 + 2\,\text{cis}\left(\frac{2n\pi}{3}\right)$

$n = 0$ leads to no solution; reject.

$n = 1: z\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\text{i}\right) = \text{i}\sqrt{3} \Rightarrow 3z = \text{i}\sqrt{3}\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\text{i}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\text{i}$

$n = -1: z\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\text{i}\right) = -\text{i}\sqrt{3} \Rightarrow 3z = -\text{i}\sqrt{3}\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\text{i}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\text{i}$

Solutions:

$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}$

23

$|z^4| = \sqrt{2^2 + (2\sqrt{3})^2} = 4 = 2^2$

$\text{Re}\,(z^4) > 0$ so $\arg(z^4) = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$z^4 = 4\,\text{cis}\left(\frac{\pi}{3} + 2n\pi\right)$ for $n \in \mathbb{Z}$

Then $z = \sqrt{2}\,\text{cis}\left(\frac{\pi + 6n\pi}{12}\right)$

$z = \sqrt{2}\text{e}^{\frac{\text{i}\pi}{12}}, \sqrt{2}\text{e}^{\frac{7\text{i}\pi}{12}}, \sqrt{2}\text{e}^{-\frac{5\text{i}\pi}{12}}, \sqrt{2}\text{e}^{-\frac{11\text{i}\pi}{12}}$

24

$|z^5| = \sqrt{16^2 + (-16\sqrt{3})^2} = 32 = 2^5$

$\text{Re}\,(z^5) > 0$ so $\arg(z^5) = \arctan\left(\frac{-16\sqrt{3}}{16}\right) = -\frac{\pi}{3}$

$z^5 = 2^5\,\text{cis}\left(-\frac{\pi}{3} + 2n\pi\right)$ for $n \in \mathbb{Z}$

Then $z = 2\,\text{cis}\left(\frac{-\pi + 6n\pi}{15}\right)$

$z = 2\text{e}^{\frac{5\text{i}\pi}{15}}, 2\text{e}^{\frac{11\text{i}\pi}{15}}, 2\text{e}^{\frac{17\text{i}\pi}{15}}, 2\text{e}^{\frac{23\text{i}\pi}{15}}, 2\text{e}^{\frac{29\text{i}\pi}{15}}$

25

$|z^6| = \sqrt{(4\sqrt{2})^2 + (-4\sqrt{2})^2} = 8 = 2^3$

$\text{Re}\,(z^6) > 0$ so $\arg(z^5) = \arctan\left(\frac{-4\sqrt{2}}{4\sqrt{2}}\right) = -\frac{\pi}{4}$

$z^6 = 2^3\,\text{cis}\left(-\frac{\pi}{4} + 2n\pi\right)$ for $n \in \mathbb{Z}$

Then $z = \sqrt{2}\,\text{cis}\left(\frac{-\pi + 8n\pi}{24}\right)$

z
$= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{24}\right), \sqrt{2}\,\text{cis}\left(-\frac{9\pi}{24}\right), \sqrt{2}\,\text{cis}\left(-\frac{17\pi}{24}\right), \sqrt{2}\,\text{cis}\left(\frac{7\pi}{24}\right), \sqrt{2}\,\text{cis}\left(\frac{15\pi}{24}\right), \sqrt{2}\,\text{cis}\left(\frac{23\pi}{24}\right)$

26

$n = 4$, since there are four points evenly spread about the origin to form a square.
The first vertex has corresponding complex number $3 + 3\text{i}$
$z^4 = (3 + 3\text{i})^4 = (18\text{i})^2 = -324$
$w = -324$

27 **a**

$\omega^5 = 1$

$$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4)$$
$$= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 - (1 + \omega + \omega^2 + \omega^3 + \omega^4)$$
$$= \omega + \omega^2 + \omega^3 + \omega^4 + 1 - (1 + \omega + \omega^2 + \omega^3 + \omega^4)$$
$$= 0$$

Since $\omega \neq 1$, it follows that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

b

$\omega^3 = \text{cis}\left(\frac{6\pi}{5}\right) = \text{cis}\left(-\frac{4\pi}{5}\right) = (\omega^2)^*$

$\omega^4 = \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(-\frac{2\pi}{5}\right) = \omega^*$

So $\text{Re}(\omega^3) = \text{Re}(\omega^2)$ and $\text{Re}(\omega^4) = \text{Re}\,(\omega)$

From part **a**, $\text{Re}(\omega + \omega^2 + \omega^3 + \omega^4) = \text{Re}(-1) = -1$

$2\big(\text{Re}(\omega) + \text{Re}(\omega^2)\big) = -1$

$\text{Re}(\omega) + \text{Re}(\omega^2) = -\frac{1}{2}$

c

$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$

Using double angle formula:

$\cos\left(\frac{2\pi}{5}\right) + 2\cos^2\left(\frac{2\pi}{5}\right) - 1 = -\frac{1}{2}$

$2\cos^2\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - \frac{1}{2} = 0$

Quadratic formula gives

$\cos\left(\frac{2\pi}{5}\right) = \frac{-1 \pm \sqrt{1+4}}{4} = \frac{-1 \pm \sqrt{5}}{4}$

$\cos\left(\frac{2\pi}{5}\right) > 0$ so $\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$

28 **a**

$z^3 = -1 = \text{cis}(\pi + 2n\pi)$ for $n \in \mathbb{Z}$

$z = \text{cis}\left(\frac{\pi + 2n\pi}{3}\right) = -1, \text{cis}\left(\pm\frac{\pi}{3}\right)$

b

$(x+2)^3 = x^3 + 6x^2 + 12x + 8$

c)

$(z+2)^3 + 1 = 0$

$(z+2)^3 = -1$

$z + 2 = -1$ or $\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}\right)$

$z = -3$ or $\left(-\frac{3}{2} \pm \frac{\sqrt{3}}{2}\text{i}\right)$

29 **a**

$z^4 = -4 = 4\,\text{cis}(\pi + 2n\pi)$ for $n \in \mathbb{Z}$

$z = \sqrt{2}\,\text{cis}\left(\frac{\pi + 2n\pi}{4}\right)$

$= \sqrt{2}\left(\pm\left(\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\text{i}\right)\right)$

$= 1 \pm \text{i}, -1 \pm \text{i}$

b

$z^4 = -4(z-1)^4$

$z = (z-1)\times\omega$ where $\omega = 1 \pm \mathrm{i}$ or $-1 \pm \mathrm{i}$

$z = -\dfrac{\omega}{1-\omega} = \dfrac{\omega}{\omega - 1}$

$z = \dfrac{1+\mathrm{i}}{\mathrm{i}}$ or $\dfrac{1-\mathrm{i}}{-\mathrm{i}}$ or $\dfrac{-1+\mathrm{i}}{-2+\mathrm{i}}$ or $\dfrac{-1-\mathrm{i}}{-2-\mathrm{i}}$

$= 1-\mathrm{i}$ or $1+\mathrm{i}$ or $\dfrac{(-1+\mathrm{i})(-2-\mathrm{i})}{(-2+\mathrm{i})(-2-\mathrm{i})}$ or $\dfrac{(-1-\mathrm{i})(-2+\mathrm{i})}{(-2-\mathrm{i})(-2+\mathrm{i})}$

$= 1-\mathrm{i}$ or $1+\mathrm{i}$ or $\dfrac{3-\mathrm{i}}{5}$ or $\dfrac{3+\mathrm{i}}{5}$

30 **a**

$\left|-32\sqrt{2} + \mathrm{i}\,32\sqrt{2}\right| = \sqrt{\left(-32\sqrt{2}\right)^2 + \left(32\sqrt{2}\right)^2} = 64 = 4^3$

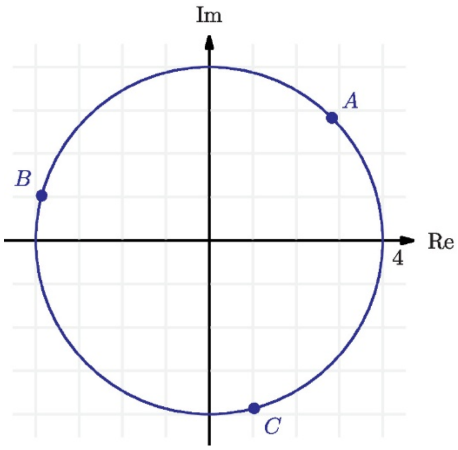
$\mathrm{Re}\left(-32\sqrt{2} + \mathrm{i}\,32\sqrt{2}\right) < 0$ so $\arg\left(-32\sqrt{2} + \mathrm{i}\,32\sqrt{2}\right) = \pi + \arctan\left(\dfrac{32\sqrt{2}}{-32\sqrt{2}}\right) = \dfrac{3\pi}{4}$

So $z^3 = 4^3 \operatorname{cis}\left(\dfrac{3\pi}{4} + 2n\pi\right)$ for $n \in \mathbb{Z}$

Then $z = 4\operatorname{cis}\left(\dfrac{3\pi + 8n\pi}{12}\right)$

$= 4\operatorname{cis}\left(\dfrac{3\pi}{12}\right), 4\operatorname{cis}\left(\dfrac{11\pi}{12}\right), 4\operatorname{cis}\left(\dfrac{19\pi}{12}\right)$

b



c

Let $\alpha = 4\operatorname{cis}\left(\frac{3\pi}{12}\right)$ and $\beta = 4\operatorname{cis}\left(\frac{11\pi}{12}\right) = \alpha\operatorname{cis}\left(\frac{2\pi}{3}\right)$, the values corresponding to points A and B, so $\alpha^3 = -32\sqrt{2} + 32\sqrt{2}\mathrm{I}$.

$w = \frac{1}{2}(\alpha+\beta) = \frac{\alpha}{2}\left(1 + \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)$

$w^3 = \dfrac{\alpha^3}{8}\left(1 + 3\operatorname{cis}\left(\dfrac{2\pi}{3}\right) + 3\operatorname{cis}\left(\dfrac{4\pi}{3}\right) + \operatorname{cis}\left(\dfrac{6\pi}{3}\right)\right)$

$= \dfrac{-32\sqrt{2} + 32\sqrt{2}\mathrm{i}}{8}(1 + 3\omega + 3\omega^2 + 1)$

Where ω is the first of the three roots of unity

But $\omega + \omega^2 + 1 = 0$ so $2 + 3\omega + 3\omega^2 = -1$

$w^3 = 4\sqrt{2} - 4\sqrt{2}\mathrm{i}$

Exercise 4E

1 **a**

Using binomial theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

$\mathrm{Re}\,(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta$

b

But by De Moivre's theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$

$$\begin{aligned}\mathrm{Re}\,(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

2 **a**

Using binomial theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^4$

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$

$\mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^4 = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

b

But by De Moivre's theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$

$$\begin{aligned}\mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^4 = \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\\ &= 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)\\ &= 4\sin\theta\cos\theta\,(1-2\sin^2\theta)\\ &= 4\cos\theta\,(\sin\theta - 2\sin^3\theta)\end{aligned}$$

3

$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \quad (1)$

So $\mathrm{e}^{-\mathrm{i}\theta} = \cos(-\theta) + \mathrm{i}\sin(-\theta) = \cos\theta - \mathrm{i}\sin\theta \quad (2)$

a $(1) + (2)\colon 2\cos\theta = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} \Rightarrow \cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2}$

b $(1) - (2)\colon 2\mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} \Rightarrow \sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}$

4 **a**

Using binomial theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^5$

$$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$$

So $\mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^5 = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

But by De Moivre's theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

$$\begin{aligned}\text{Then }\mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^5 = \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta\\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

b

If $\sin 5\theta = -4\sin^5\theta$

Then $20\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 0$

$5\sin\theta\,(4\sin^4\theta - 4\sin^2\theta + 1) = 0$

$5\sin\theta\,(2\sin^2\theta - 1)^2 = 0$

$\sin\theta = 0 \text{ or } \pm\frac{1}{\sqrt{2}}$

Worked solutions

5 **a**

If $z = \cos\theta + \mathrm{i}\sin\theta$ then $z^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ by De Moivre's theorem

So $z^n + z^{-n} = \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = 2\cos(n\theta)$

b

$$\begin{aligned}\cos^4\theta &= (\cos\theta)^4\\ &= \left(\frac{z+z^{-1}}{2}\right)^4\\ &= \frac{1}{16}(z^4+z^{-4}+4z^2+4z^{-2}+6)\\ &= \frac{1}{8}\left(\frac{z^4+z^{-4}}{2}+4\frac{(z^2+z^{-2})}{2}+3\right)\\ &= \frac{1}{8}(\cos(4\theta)+4\cos(2\theta)+3)\end{aligned}$$

c

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\cos^4\theta\ \mathrm{d}\theta &= \frac{1}{8}\int_0^{\frac{\pi}{4}}\cos(4\theta)+4\cos(2\theta)+3\ \mathrm{d}\theta\\ &= \frac{1}{8}\left[\frac{1}{4}\sin(4\theta)+2\sin(2\theta)+3\theta\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{8}\left(\left(0+2+\frac{3\pi}{4}\right)-0\right)\\ &= \frac{3\pi+8}{32}\end{aligned}$$

6 **a**

If $z = \cos\theta + \mathrm{i}\sin\theta$ then $z^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ by De Moivre's theorem

So $z^n - z^{-n} = \cos(n\theta) + \mathrm{i}\sin(n\theta) - (\cos(-n\theta) + \mathrm{i}\sin(-n\theta)) = 2\mathrm{i}\sin(n\theta)$

b

$$\begin{aligned}\sin^5\theta &= (\sin\theta)^5\\ &= \left(\frac{z-z^{-1}}{2\mathrm{i}}\right)^5\\ &= \frac{1}{32\mathrm{i}}(z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5})\\ &= \frac{1}{16}\left(\frac{z^5-z^{-5}}{2\mathrm{i}}-5\frac{(z^3-z^{-3})}{2\mathrm{i}}+10\frac{(z-z^{-1})}{2\mathrm{i}}\right)\\ &= \frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)\end{aligned}$$

c

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sin^5\theta\ \mathrm{d}\theta &= \frac{1}{16}\int_0^{\frac{\pi}{2}}\sin 5\theta-5\sin 3\theta+10\sin\theta\ \mathrm{d}\theta\\ &= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta+\frac{5}{3}\cos 3\theta-10\cos\theta\right]_0^{\frac{\pi}{2}}\\ &= \frac{1}{16}\left(0-\left(-\frac{1}{5}+\frac{5}{3}-10\right)\right)\\ &= \frac{1}{16}\left(\frac{128}{15}\right)\end{aligned}$$

$$= \frac{8}{15}$$

7 **a**

If $z = \cos\theta + \mathrm{i}\sin\theta$ then $z^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ by De Moivre's theorem

So $z^n + z^{-n} = \cos(n\theta) + \mathrm{i}\sin(n\theta) + (\cos(-n\theta) + \mathrm{i}\sin(-n\theta)) = 2\cos(n\theta)$

$$32\cos^6\theta = 32\left(\frac{z+z^{-1}}{2}\right)^6$$
$$= \frac{1}{2}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$
$$= \frac{z^6+z^{-6}}{2} + 6\frac{(z^4+z^{-4})}{2} + 15\frac{z^2+z^{-2}}{2} + 10$$
$$= \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

$A = 6, B = 15, C = 10$

b

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \ \mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10) \ \mathrm{d}\theta$$
$$= \frac{1}{32}\left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{32}\big((0 + 5\pi) - (0)\big)$$
$$= \frac{5\pi}{32}$$

8 **a**

Using binomial theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^5$

$$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$$

So $\mathrm{Re}\,(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

But by De Moivre's theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

Then $\mathrm{Re}\,(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

b

$16x^5 - 20x^3 + 5x = -1$

First, looking for roots with values $-1 \le x \le 1$ so that $x = \cos\theta$ for some $\theta \in [0,2\pi)$

By part **a**,

$\cos 5\theta = -1$

So $5\theta = \pi + 2n\pi$ for $n \in \mathbb{Z}$

$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

Then $x = \cos\theta = 0.809, -0.309, -1$

Considering roots outside this interval:

$$\frac{\mathrm{d}}{\mathrm{d}x}(16x^5 - 20x^3 + 5x) = 80x^4 - 60x^2 + 5 > 0 \text{ for } |x| > 1$$

So the curve cannot turn and pass again through -1.
The three roots given are therefore the only roots to the equation
$16x^5 - 20x^3 + 5x + 1 = 0$

There are several ways to argue that only roots which can be expressed as $x = \cos\theta$ need to be considered, but to have a rigorous answer some such argument should be made.

9 **a**

Using binomial theorem:

$$(z + z^{-1})^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$$
$$(z - z^{-1})^6 = z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}$$

b

If $z = \operatorname{cis}\theta$ then by De Moivre's theorem, $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$

$$\begin{aligned}\cos^6\theta + \sin^6\theta &= \left(\frac{z + z^{-1}}{2}\right)^6 + \left(\frac{z - z^{-1}}{2\mathrm{i}}\right)^6 \\ &= \left(\frac{z + z^{-1}}{2}\right)^6 - \left(\frac{z - z^{-1}}{2}\right)^6 \\ &= \frac{1}{64}(12z^4 + 40 + 12z^{-4}) (\text{by part } \mathbf{a}) \\ &= \frac{1}{8}\left(3\frac{z^4 + z^{-4}}{2} + 5\right) \\ &= \frac{1}{8}(3\cos 4\theta + 5)\end{aligned}$$

10 **a**

$$\begin{aligned}(\operatorname{cis} x)^5 &= \cos^5 x + 5\mathrm{i}\cos^4 x \sin x - 10\cos^3 x \sin^2 x - 10\mathrm{i}\cos^2 x \sin^3 x \\ &\quad + 5\cos x \sin^4 x + \mathrm{i}\sin^5 x \\ &= (\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x) \\ &\quad + \mathrm{i}(5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x)\end{aligned}$$

b

By De Moivre's theorem, $(\operatorname{cis} x)^5 = \cos 5x + \mathrm{i}\sin 5x$
Equating imaginary parts,

$$\begin{aligned}\sin 5x &= 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x \\ &= \sin x\,(5\cos^4 x - 10\cos^2 x\,(1 - \cos^2 x) + (1 - \cos^2 x)^2) \\ &= \sin x\,(16\cos^4 x - 12\cos^2 x + 1)\end{aligned}$$

c

From part **b:**

$$\lim_{x\to 0}\frac{\sin 5x}{\sin x} = \lim_{x\to 0}(16\cos^4 x - 12\cos^2 x + 1) = 16 - 12 + 1 = 5$$

11 **a**

$$\begin{aligned}\cos 3\theta &= Re\,(\operatorname{cis} 3\theta) \\ &= \mathrm{Re}\,((\operatorname{cis}\theta)^3) (\text{by De Moivre's theorem}) \\ &= \mathrm{Re}\,(\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta) \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

b

If $x = \cos\theta$ then $8x^3 - 6x - 1 = 2\cos 3\theta - 1$ by part **a**

$2\cos 3\theta - 1 = 0$

$\cos 3\theta = \frac{1}{2}$

$3\theta = \pm\frac{\pi}{3} + 2n\pi$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}$ or $\frac{7\pi}{9}$

The original cubic can have at most 3 roots; since the above finds three different solutions these must be the three roots.

The roots are $x = \cos\left(\frac{\pi}{9}\right), \cos\left(\frac{5\pi}{9}\right)$ or $\cos\left(\frac{7\pi}{9}\right)$

12 **a**

By the binomial theorem:

$(\cos\theta + \mathrm{i}\sin\theta)^4$

$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$

$\mathrm{Re}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\mathrm{Im}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

b

By the De Moivre theorem, $\mathrm{Re}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right) = \cos 4\theta$ and $\mathrm{Im}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right) = \sin 4\theta$

So $\tan 4\theta = \frac{\mathrm{Im}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right)}{\mathrm{Re}\left((\cos\theta + \mathrm{i}\sin\theta)^4\right)}$

$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Dividing numerator and denominator by $\cos^4\theta$:

$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

c

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$1 - 6x^2 + x^4 = 4x - 4x^3$

$\frac{1 - 6x^2 + x^4}{4x - 4x^3} = 1$

Let $x = \tan\theta$

Then, using part **b**: $\tan 4\theta = 1$

$4\theta = \frac{\pi}{4} + n\pi = \frac{(1+4n)\pi}{4}$ for $n \in \mathbb{Z}$

$\theta = \frac{(1+4n)\pi}{16} = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

The four solutions to the quartic are

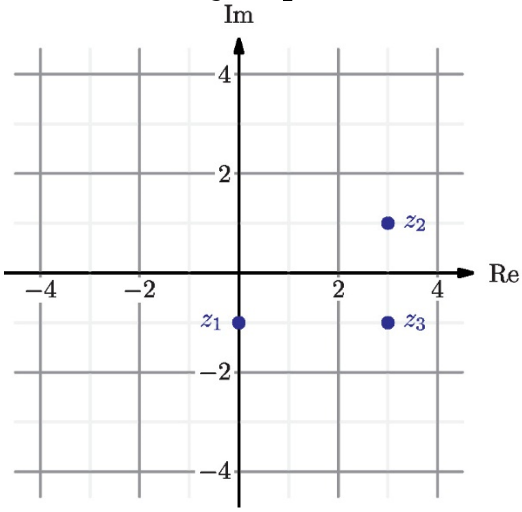
$x = \tan\left(\frac{\pi}{16}\right), \tan\left(\frac{5\pi}{16}\right), \tan\left(\frac{9\pi}{16}\right)$ or $\tan\left(\frac{13\pi}{16}\right)$

Mixed Practice

1 **a** $z_1 = \frac{1}{i} = -i$

b $z_2 = (1+i)(2-i) = 3+i$

c $z_3 = z_2^* = 3-i$



2

$x^2 - 2x + 2 = (x-1)^2 + 1$

$= (x - 1 + i)(x - 1 - i)$

$x = 1 \pm i$

3

$x^2 - 6x + 12 = (x-3)^2 + 3$

$= \left(x - 3 + i\sqrt{3}\right)\left(x - 3 - i\sqrt{3}\right)$

$x = 3 \pm i\sqrt{3}$

4

$$z = \frac{1+i}{1+2i}$$

$$= \frac{1+i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{3-i}{5}$$

$$= \frac{3}{5} - \frac{1}{5}i$$

$$z^* = \frac{3}{5} + \frac{1}{5}i$$

5

For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $1 + 2i$ is a root, $1 - 2i$ is also a root.

$(x - 1 - 2i)(x - 1 + 2i) = 0$

$x^2 - 2x + 5 = 0$

$b = -2, c = 5$

6

$\frac{z}{z+\mathrm{i}} = 1 + 2\mathrm{i}$
$z = (1 + 2\mathrm{i})(z + \mathrm{i})$
$z = z + 2\mathrm{i}z - 2 + \mathrm{i}$
$2\mathrm{i}z = 2 - \mathrm{i}$
$z = -\frac{1}{2} - \mathrm{i}$

7

$z + \mathrm{i} = 2z^*$
Let $z = x + \mathrm{i}y$ where $x = \mathrm{Re}\,(z), y = \mathrm{Im}\,(z)$ so $z^* = x - \mathrm{i}y$
$x + (y + 1)\mathrm{i} = 2x - 2\mathrm{i}y$
Comparing real and imaginary parts:
$x = 2x$ so $x = 0$
$y + 1 = -2y$ so $y = -\frac{1}{3}$
$z = -\frac{1}{3}\mathrm{i}$

8

$z + 4\mathrm{i} = \mathrm{i}z$
Let $z = x + \mathrm{i}y$ where $x = \mathrm{Re}\,(z), y = \mathrm{Im}\,(z)$
$x + y\mathrm{i} + 4\mathrm{i} = -y + \mathrm{i}x$
Comparing real and imaginary parts:
$x = -y \qquad (1)$
$y + 4 = x \qquad (2)$
Substituting (1) into (2):
$y + 4 = -y$
$y = -2, x = 2$
$z = 2 - 2\mathrm{i}$

9 **a**

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$
Since $\mathrm{Re}\, z > 0$, $\arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

b

$z = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$ so $z^6 = 8\,\mathrm{cis}\left(\frac{3\pi}{2}\right) = -8\mathrm{i}$
$w = \mathrm{cis}\left(\frac{3\pi}{5}\right)$ so $w^5 = \mathrm{cis}(3\pi) = -1$
Then $z^6 w^5 = 8\mathrm{i}$

10

$|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$
Since $\mathrm{Re}\, z > 0$, $\arg z = \arctan\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$
Then
$|(z^*)^3| = \left(2\sqrt{2}\right)^3 = 16\sqrt{2}$
$\arg((z^*)^3) = \frac{3\pi}{4}$

11 **a**

Comparing real and imaginary parts:

$p = 3$

$4q = 2$ so $q = \frac{1}{2}$

b

If $p = a + \mathrm{i}b$ and $q = p^* = a - \mathrm{i}b$

$a\mathrm{i} - b + 4a - 4ib = 2 + 3\mathrm{i}$

Comparing real and imaginary parts:

$4a - b = 2 \qquad (1)$

$a - 4b = 3 \qquad (2)$

$4(1) - (2)$: $15a = 5$ so $a = \frac{1}{3}$

(1): $b = 4a - 2 = -\frac{2}{3}$

$p = \frac{1}{3} - \frac{2}{3}\mathrm{i}, q = \frac{1}{3} + \frac{2}{3}\mathrm{i}$

12

Let $z^3 = -1 = \text{cis}(\pi + 2n\pi)$ for $n \in \mathbb{Z}$

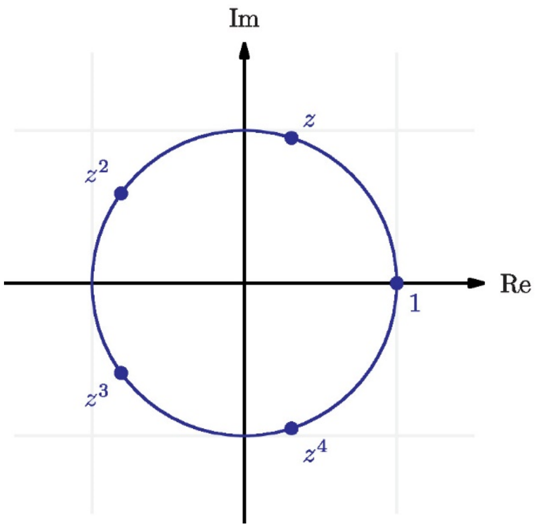
Then $z = \text{cis}\left(\frac{\pi + 2n\pi}{3}\right)$

$= \text{cis}\left(\frac{\pi}{3}\right), \text{cis}(\pi)$ or $\text{cis}\left(\frac{5\pi}{3}\right)$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$ or -1

13 **a** $1, z_1 = e^{\frac{2\pi \mathrm{i}}{5}}, z_2 = z_1^2 = e^{\frac{4\pi \mathrm{i}}{5}}, z_3 = z_1^3 = e^{\frac{6\pi \mathrm{i}}{5}}, z_4 = z_1^4 = e^{\frac{8\pi \mathrm{i}}{5}}$

b



14 **a** $\text{cis}\left(\frac{\pi}{2}\right) \div \text{cis}\left(\frac{\pi}{6}\right) = \text{cis}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

b $\text{cis}\left(\frac{\pi}{2}\right) - \text{cis}\left(\frac{\pi}{6}\right) = \mathrm{i} - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

15

$$\frac{1}{a+\mathrm{i}} = \frac{1}{a+\mathrm{i}} \times \frac{a-\mathrm{i}}{a-\mathrm{i}} = \frac{a}{a^2+1} - \frac{1}{a^2+1}\mathrm{i}$$

16

$$\frac{1}{z+\mathrm{i}} = -\frac{2}{z-\mathrm{i}}$$

Taking reciprocals on both sides:

$$z+\mathrm{i} = -\frac{z-\mathrm{i}}{2}$$

$$\frac{3}{2}z = -\frac{1}{2}\mathrm{i}$$

$$z = -\frac{1}{3}\mathrm{i}$$

17

Let $z = x + \mathrm{i}y$ for real x, y

Then $z^* = x - \mathrm{i}y$

$z + z^* = 2x = 8$ so $x = 4$

$z - z^* = 2\mathrm{i}y = 6\mathrm{i}$ so $y = 3$

$z = 4 + 3\mathrm{i}$

18

Let $\mathrm{f}(x) = x^3 - x^2 + x - 1$

By inspection, $f(1) = 0$ so $(x-1)$ is a factor of $\mathrm{f}(x)$.

$\mathrm{f}(x) = (x-1)(ax^2 + bx + c) = x^3 - x^2 + x - 1$

Expanding and comparing coefficients:

$x^3: a = 1$

$x^2: b - a = -1$ so $b = 0$

$x^1: c - b = 1$ so $c = 1$

$x^0: -c = -1$ is consistent with this

$\mathrm{f}(x) = (x-1)(x^2+1) = (x-1)(x+\mathrm{i})(x-\mathrm{i})$

The roots are $1, \pm\mathrm{i}$

19

Let $\mathrm{f}(x) = x^3 - 5x^2 + 7x + 13$

By inspection, $f(-1) = 0$ so $(x+1)$ is a factor of $\mathrm{f}(x)$.

$\mathrm{f}(x) = (x+1)(ax^2 + bx + c) = x^3 - 5x^2 + 7x + 13$

Expanding and comparing coefficients:

$x^3: a = 1$

$x^2: b + a = -5$ so $b = -6$

$x^1: c + b = 7$ so $c = 13$

$x^0: c = 13$ is consistent with this

$\mathrm{f}(x) = (x+1)(x^2 - 6x + 13) = (x+1)(x-3+2\mathrm{i})(x-3-2\mathrm{i})$

The roots are $-1, 3 \pm 2\mathrm{i}$

20

$$|zw| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2,$$

$$\text{Since } \mathrm{Re}(zw) < 0, \arg(zw) = \pi + \arctan\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$$

$\left|\frac{z}{w}\right| = \sqrt{\left(-\frac{1}{2}\right)^2} = \frac{1}{2},$

$\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$

$|zw| \times \left|\frac{z}{w}\right| = |z^2| = 1 \text{ so } |z| = 1$

$\arg(zw) + \arg\left(\frac{z}{w}\right) = \arg(z^2) = \frac{\pi}{3} \text{ so } \arg z = \frac{\pi}{6}$

21

$\frac{2}{2+\mathrm{i}} - \frac{1}{b+\mathrm{i}} = \frac{2(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} - \frac{b-\mathrm{i}}{(b+\mathrm{i})(b-\mathrm{i})}$

$= \frac{4-2\mathrm{i}}{5} - \frac{b-\mathrm{i}}{b^2+1}$

If this is a real value then its imaginary part equals zero.

$-\frac{2}{5} + \frac{1}{b^2+1} = 0$

$2(b^2+1) = 5$

$b^2 + 1 = \frac{5}{2}$

$b = \pm\sqrt{\frac{3}{2}}$

22

Let $z = 2 + \mathrm{i}y$ for $y \in \mathbb{R}$

Then $z^2 = 4 - y^2 + 4\mathrm{i}y$

$4 - y^2 = 3 \text{ so } y = \pm 1$

$z = 2 \pm \mathrm{i}$

23

Let $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$

$z + |z| = 18 + 12\mathrm{i}$

Comparing real and imaginary parts:

$x + \sqrt{x^2 + y^2} = 18$

$y = 12$

$x^2 + 144 = (18 - x)^2 = x^2 + 324 - 36x$

$36x = 180$

$x = 5$

$z = 5 + 12\mathrm{i}$

24 **a**

$|z - 4| = 2|z - 1|$

Let $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$

$\sqrt{(x-4)^2 + y^2} = 2\sqrt{(x-1)^2 + y^2}$

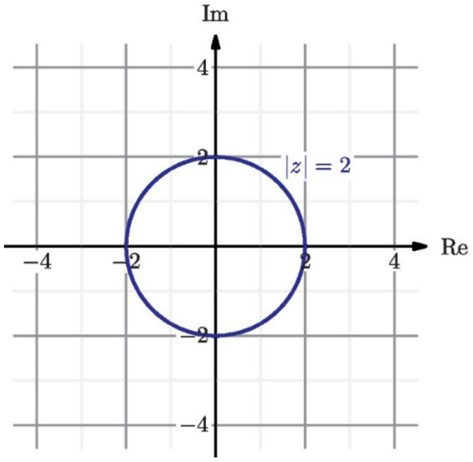
$x^2 - 8x + 16 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$3x^2 + 3y^2 = 12$

$x^2 + y^2 = 4$

$\sqrt{x^2 + y^2} = |z| = 2$

b



25

Let $\mathrm{f}(z) = z^3 - 11z^2 + 43z - 65$

For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $3 - 2\mathrm{i}$ is a root, $3 + 2\mathrm{i}$ is also a root. The final root must be real, $z = k$.

$$\begin{aligned}\mathrm{f}(z) &= (z - 3 + 2\mathrm{i})(z - 3 - 2\mathrm{i})(z - k)\\ &= (z^2 - 6z + 13)(z - k)\\ &= z^3 - 11z^2 + 43z - 65\end{aligned}$$

Comparing coefficients:

$z^3: 1 = 1$

$z^2: -6 - k = -11$ so $k = 5$

$z^1: 13 + 6k = 43$ is consistent with $k = 5$

$z^0: -13k = -65$ is consistent with $k = 5$

The roots are $3 \pm 2\mathrm{i}, 5$

26 **a** If $z = \mathrm{i}$ then $\mathrm{Re}\,(z) = 0$ but $\mathrm{Re}\,(z^2) = \mathrm{Re}(-1) = -1 \neq 0^2$

b $\mathrm{Re}(z^2) = \left(\mathrm{Re}(z)\right)^2 - \left(\mathrm{Im}\,(z)\right)^2$

So if $\mathrm{Re}(z^2) = \left(\mathrm{Re}(z)\right)^2$ then $\left(\mathrm{Im}\,(z)\right)^2 = 0$ so $\mathrm{Im}\,(z) = 0$

27 **ai**

$z^3 = -8 = 2^3\,\mathrm{cis}(\pi + 2n\pi)$ for $n \in \mathbb{Z}$

$$z = 2\,\mathrm{cis}\left(\frac{\pi + 2n\pi}{3}\right) = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right), 2\,\mathrm{cis}(\pi), 2\mathrm{cis}\left(\frac{5\pi}{3}\right)$$

aii

$z = 1 \pm \mathrm{i}\sqrt{3}$ or -2

b

The triangle has (vertical) base length $2\sqrt{3}$ and (horizontal) altitude 3 so has area

$\frac{1}{2} \times 2\sqrt{3} \times 3 = 3\sqrt{3}$

28

$2|z| = |z + 3|$

$2\sqrt{x^2 + y^2} = \sqrt{(x + 3)^2 + y^2}$

$4x^2 + 4y^2 = x^2 + 6x + 9 + y^2$

$3x^2 + 3y^2 - 6x = 9$

$x^2 + y^2 - 2x = 3$

This form is fine as a final answer, since the question asks for the relationship between the variables.

You could continue to complete the square

$(x-1)^2+y^2=4$

or find y in terms of x

$y=\pm\sqrt{4-(x-1)^2}$

but this is not necessary as the question is phrased.

29

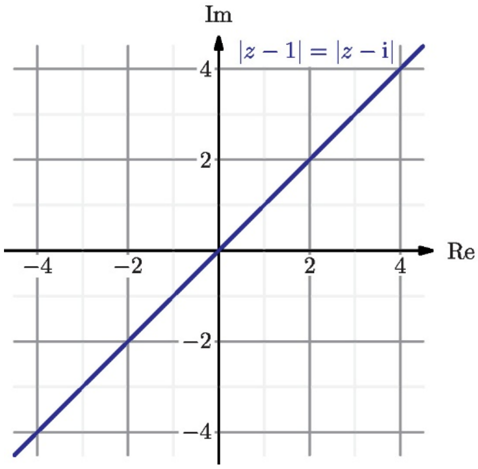
$$|z+pw|^2 = (1+2p)^2+(1-p)^2$$
$$= 5p^2+2p+2$$

Completing the square:

$$|z+pw|^2 = 5\left(p^2+\frac{2}{5}p\right)+2$$
$$= 5\left(\left(p+\frac{1}{5}\right)^2-\frac{1}{25}\right)+2$$
$$= 5\left(p+\frac{1}{5}\right)^2+\frac{9}{5}$$

$|z+pw|^2$ has minimum value $\frac{9}{5}$ so $|z+pw|$ has minimum value $\frac{3}{\sqrt{5}}$

30 If $|z-1|=|z-\text{i}|$ then z lies on the perpendicular bisector of 1 and i



31 **a**

Let $z=\text{cis}\,\theta$

By De Moivre's theorem, $z^n=\text{cis}\,n\theta$

Using binomial theorem to expand $(\text{cis}\,\theta)^5$

$$\text{Re}\,(z^n) = \cos 5\theta$$
$$= \text{Re}\,(\cos^5\theta+5\text{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10\text{i}\cos^2\theta\sin^3\theta$$
$$+5\cos\theta\sin^4\theta+\text{i}\sin^5\theta$$
$$= \cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$= \cos^5\theta-10\cos^3\theta\,(1-\cos^2\theta)+5\cos\theta\,(1-\cos^2\theta)^2$$
$$= 16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

b

If $\cos 5\theta = 5\cos\theta$:

$16\cos^5\theta-20\cos^3\theta=0$

$4\cos^3\theta\,(4\cos^2\theta-5)=0$

$\cos\theta = 0$ or $\cos^2\theta = \dfrac{5}{4}$ (reject, outside the range of $\cos\theta$)

$\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

32 **a**

$$zw = (1+\mathrm{i})(1+\mathrm{i}\sqrt{3})$$
$$= (1-\sqrt{3}) + \mathrm{i}(1+\sqrt{3})$$

b

$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

Since Re $(z) > 0$, $\arg z = \arctan\left(\dfrac{1}{1}\right) = \dfrac{\pi}{4}$

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$$

$$|w| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

Since Re $(w) > 0$, $\arg w = \arctan\left(\dfrac{\sqrt{3}}{1}\right) = \dfrac{\pi}{3}$

$$w = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

Then $|zw| = 2\sqrt{2}$ and $\arg(zw) = \dfrac{\pi}{4} + \dfrac{\pi}{3} = \dfrac{7\pi}{12}$

c

Comparing the results from parts **a** and **b**

$$\sin\left(\frac{7\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

33 **a**

$$|\sqrt{3}+\mathrm{i}| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

Re $(\sqrt{3}+\mathrm{i}) > 0$ so $\arg(\sqrt{3}+\mathrm{i}) = \arctan\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

b

$$(\sqrt{3}+\mathrm{i})^7 = \left(2\,\mathrm{cis}\left(\frac{\pi}{6}\right)\right)^7 = 2^7\,\mathrm{cis}\left(\frac{7\pi}{6}\right)$$
$$(\sqrt{3}-\mathrm{i})^7 = \left(2\,\mathrm{cis}\left(-\frac{\pi}{6}\right)\right)^7 = 2^7\,\mathrm{cis}\left(-\frac{7\pi}{6}\right)$$
$$(\sqrt{3}+\mathrm{i})^7 + (\sqrt{3}-\mathrm{i})^7 = 2^7\left(2\cos\left(\frac{7\pi}{6}\right)\right) = 128 \times 2 \times \left(-\frac{\sqrt{3}}{2}\right) = -128\sqrt{3}$$

34 **a**

$z^3 = 8 = 2^3\,\mathrm{cis}(2n\pi)$ for $n \in \mathbb{Z}$

$$z = 2\,\mathrm{cis}\left(\frac{2n\pi}{3}\right)$$
$$= 2\,\mathrm{cis}(0) \text{ or } 2\,\mathrm{cis}\left(\frac{2\pi}{3}\right) \text{ or } 2\,\mathrm{cis}\left(\frac{4\pi}{3}\right)$$
$$= 2 \text{ or } -1 \pm \mathrm{i}\sqrt{3}$$

b

$(z+2)^3 = 8z^3$

$\dfrac{z+2}{z} = w$ where $w = 2$ or $-1 \pm i\sqrt{3}$

$z + 2 = zw$

$z = \dfrac{2}{w-1}$

$= 2$ or $\dfrac{2}{-2 \pm i\sqrt{3}}$

$= 2$ or $\dfrac{2(-2 \mp i\sqrt{3})}{7}$

$= 2$ or $-\dfrac{4}{7} \pm \dfrac{2\sqrt{3}}{7}i$

35 **a**

$e^{i\theta} = \cos\theta + i\sin\theta$ and so $e^{-i\theta} = \cos\theta - i\sin\theta$

Then $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

So $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

b

Using binomial theorem:

$(x + x^{-1})^4 = x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4}$

c

If $x = e^{i\theta}$ then $x^{-1} = e^{-i\theta}$ and $(x + x^{-1}) = 2\cos\theta$

By De Moivre's theorem, $(x^n + x^{-n}) = 2\cos n\theta$

From part **b**:

$$\begin{aligned}(2\cos\theta)^4 &= x^4 + x^{-4} + 4x^2 + 4x^{-2} + 6\\ &= 2\cos 4\theta + 8\cos 2\theta + 6\end{aligned}$$

$\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

$a = \dfrac{1}{8}, b = \dfrac{1}{2}, c = \dfrac{3}{8}$

d

$$\begin{aligned}\int_0^\pi \cos^4(2x)\,dx &= \frac{1}{8}\int_0^\pi \cos 8x + 4\cos 4x + 3\ dx\\ &= \frac{1}{8}\left[\frac{1}{8}\sin 8x + \sin 4x + 3x\right]_0^\pi\\ &= \frac{1}{8}(3\pi - 0)\\ &= \frac{3\pi}{8}\end{aligned}$$

36 **a**

$$\begin{aligned}AB &= |z_1 - z_2|\\ &= \left|1 - (2-\sqrt{3})i\right|\\ &= \sqrt{1^2 + \left(2-\sqrt{3}\right)^2}\\ &= \sqrt{8 - 4\sqrt{3}}\\ &= 2\sqrt{2-\sqrt{3}}\end{aligned}$$

b

$\arg z_1 = \arctan\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$

$\arg z_2 = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

Then angle subtended at the origin between the two points in the complex plane is the difference in their arguments:

$\arg z_1 - \arg z_2 = \frac{\pi}{12}$

37 **a**

$\omega^3 = 1$ but $\omega \neq 1$

Then $(1-\omega)(1+\omega+\omega^2) = 1+\omega+\omega^2 - (\omega+\omega^2+1) = 0$

Since $\omega \neq 1$, it follows that $1+\omega+\omega^2 = 0$

b

Then $\omega + \omega^2 = -1$

$$\begin{aligned}(\omega x + \omega^2 y)(\omega^2 x + \omega y) &= \omega^3(x^2+y^2) + (\omega^2+\omega^4)xy \\ &= x^2+y^2+(\omega+\omega^2)xy \\ &= x^2+y^2-xy\end{aligned}$$

38 **a** $z = \mathrm{Re}\,(z) + \mathrm{Im}\,(z)$ and $z^* = \mathrm{Re}\,(z) - \mathrm{Im}\,(z)$

Then $z + z^* = 2\,\mathrm{Re}\,(z)$

b $(zw^*)^* = z^*w$

c From part **a**, $zw^* + (zw^*)^* = 2\,\mathrm{Re}\,(zw^*)$ so is real

39

$\mathrm{e}^{\mathrm{i}a} = \cos a + \mathrm{i}\sin a$ and $\mathrm{e}^{-\mathrm{i}a} = \cos a - \mathrm{i}\sin a$

So $\mathrm{e}^{\mathrm{i}a} + \mathrm{e}^{-\mathrm{i}a} = 2\cos a$

Then if $a = \mathrm{i}x$:

$\mathrm{e}^{\mathrm{i}a} = \mathrm{e}^{-x} \in \mathbb{R}$ for $x \in \mathbb{R}$ and $\mathrm{e}^{-\mathrm{i}a} = \mathrm{e}^{x} \in \mathbb{R}$

So $\mathrm{e}^{\mathrm{i}a} + \mathrm{e}^{-\mathrm{i}a} \in \mathbb{R}$

But $\mathrm{e}^{\mathrm{i}a} + \mathrm{e}^{-\mathrm{i}a} = 2\cos a = 2\cos \mathrm{i}x$

Hence $\cos \mathrm{i}x \in \mathbb{R}$

If you consider $\cos y$ in its Maclaurin expansion, as an even function of y, then it is also clear that $\cos \mathrm{i}x$ must be real, since it is a sum of even powers of $\mathrm{i}x$, each of which must be real for real x.

40

If $|z| = r$ and $\arg z = \theta$

$|z|z^3 = r^4 \operatorname{cis} 3\theta = -81 = 3^4 \operatorname{cis}(\pi + 2n\pi)$

$r = 3, \theta = \frac{\pi + 2n\pi}{3}$

$$\begin{aligned}\text{Then } z &= 3\operatorname{cis}\left(\frac{\pi}{3}\right), 3\operatorname{cis}(\pi) \text{ or } 3\operatorname{cis}\left(\frac{5\pi}{3}\right) \\ &= \frac{3}{2} \pm \frac{3\sqrt{3}}{2}\mathrm{i} \text{ or } -3\end{aligned}$$

41 **a**

If $z = r\operatorname{cis}\theta$ then $z^* = r\operatorname{cis}(-\theta)$

So $zz^* = r^2\operatorname{cis}(\theta-\theta) = r^2 = |z|^2$

b

If $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ for $x, y, u, v \in \mathbb{R}$ then

$|z - w|^2 = (x - u)^2 + (y - v)^2 = x^2 + y^2 + u^2 + v^2 - 2ux - 2yv$

$|z + w|^2 = (x + u)^2 + (y + v)^2 = x^2 + y^2 + u^2 + v^2 + 2ux + 2yv$

So $|z - w|^2 + |z + w|^2 = 2(x^2 + y^2 + u^2 + v^2) = 2|z|^2 + 2|w|^2$

42

$$z|z| + \frac{2}{z^*} = 3z$$

Let $z = r \operatorname{cis} \theta$

Then $|z| = r, \dfrac{1}{z^*} = \dfrac{1}{r} \operatorname{cis} \theta$

So $\mathrm{r}^2 \operatorname{cis} \theta + \dfrac{2}{r} \operatorname{cis} \theta = 3r \operatorname{cis} \theta$

$$\frac{r^3 + 2 - 3r^2}{r} \operatorname{cis} \theta = 0$$

Since $\operatorname{cis} \theta \neq 0$, $r^3 + 2 - 3r^2 = 0$

By observation, $r = 1$ is a solution of this, so $(r - 1)$ must be a factor.

$$\begin{aligned} r^3 - 3r^2 + 2 &= (r - 1)(r^2 - 2r - 2) \\ &= (r - 1)(r - 1 - \sqrt{3})(r - 1 + \sqrt{3}) \end{aligned}$$

So the solutions are $r = 1, 1 \pm \sqrt{3}$

43 **a**

$$\begin{aligned} |z + 1| &= \sqrt{(1 + \cos 2\theta)^2 + \sin^2 2\theta} \\ &= \sqrt{1 + \cos^2 2\theta + \sin^2 2\theta + 2\cos 2\theta} \\ &= \sqrt{2(1 + \cos 2\theta)} \\ &= \sqrt{4\cos^2 \theta} \\ &= 2\cos \theta \end{aligned}$$

(selecting positive root because the modulus must be positive)

b

$$\begin{aligned} \mathrm{Re}\,(z + 1) > 0 \text{ so } \arg(z + 1) &= \arctan\left(\frac{\sin 2\theta}{1 + \cos 2\theta}\right) \\ &= \arctan\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right) \\ &= \arctan(\tan\theta) \\ &= \theta \end{aligned}$$

44 **a**

Let $z = \operatorname{cis} \theta$

By De Moivre's theorem, $z^n = \operatorname{cis} n\theta$

Using binomial theorem to expand $(\operatorname{cis} \theta)^3$

$$\begin{aligned} \mathrm{Re}\,(z^3) &= \cos 3\theta \\ &= \mathrm{Re}\,(\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

b

$$\omega = \operatorname{cis}\left(\frac{2\pi}{7}\right) \neq 1$$

$\omega^7 = \operatorname{cis} 2\pi = 1$

bi

$$(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6)$$
$$=(1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6)$$
$$-(\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7)=0$$

Since $\omega \neq 1$, it follows that $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$

bii

$$\omega^6=\text{cis}\left(\frac{12\pi}{7}\right)=\text{cis}\left(-\frac{2\pi}{7}\right)=\omega^*$$

Similarly, $\omega^5=(\omega^2)^*$ and $\omega^4=(\omega^3)^*$

Then $\omega+\omega^6=2\,\text{Re}\,(\omega)=2\cos\left(\frac{2\pi}{7}\right), \omega^2+\omega^5=2\cos\left(\frac{4\pi}{7}\right), \omega^3+\omega^4$
$$=2\cos\left(\frac{6\pi}{7}\right)$$

From part **bi**, $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$

Taking the real part of this:

$$1+2\left(\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)\right)=0$$

$$\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\frac{1}{2}$$

c

Let $\cos\left(\frac{2\pi}{7}\right)=t$

Using the identity from part **a** with $\theta=\frac{2\pi}{7}$

$$\cos\left(\frac{6\pi}{7}\right)=4t^3-3t$$

Using the double angle formula,

$$\cos\left(\frac{4\pi}{7}\right)=2t^2-1$$

Substituting these into the result from part **bii**:

$$t+(2t^2-1)+(4t^3-3t)=-\frac{1}{2}$$
$$8t^3+4t^2-4t-2=-1$$
$$8t^3+4t^2-4t-1=0$$

Hence $\cos\left(\frac{2\pi}{7}\right)=t$ is a root of the given cubic.

45 **ai**

$$|z_1|=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2$$

$$|z_2|=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$$

$$|z_3|=\sqrt{(-2)^2}=2$$

$\text{Re}\,(z_1)>0$ so $\arg z_1=\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$\text{Re}\,(z_2)\leq 0$ so $\arg z_2=\pi+\arctan\left(\frac{1}{-\sqrt{3}}\right)=\frac{5\pi}{6}$

$$\arg z_3=\frac{3\pi}{2}$$

$$z_1=2\text{cis}\left(\frac{\pi}{6}\right), z_2=2\text{cis}\left(\frac{5\pi}{6}\right), z_3=2\text{cis}\left(\frac{3\pi}{2}\right)$$

aii

Since each of the three have the same modulus, and their arguments differ by $\frac{2\pi}{3}$, it follows that they all lie on a circle about the origin with radius 2, equally spread so that they represent the vertices of an equilateral triangle with centre at the origin.

aiii

It follows that if $\omega = \text{cis}\left(\frac{2\pi}{3}\right)$ is the first of the three roots of unity,

$z_2 = \omega z_1$ and $z_3 = \omega^2 z_1$

So $z_1^{3n} + z_2^{3n} = z_1^{3n}(1 + \omega^{3n}) = 2z_1^{3n}$

And $z_3^{3n} = z_1^{3n}(\omega^2)^{3n} = z_1^{3n}(\omega^3)^{2n} = z_1^{3n}$

Therefore $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$

bi $\text{cis}\left(\frac{2k\pi}{7}\right)$ for $k = 0, 1, \dots, 6$

bii

$$\text{Re}\,(1+w) > 0 \text{ so } \arg(1+w) = \arctan\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}\right)$$

$$= \arctan\left(\frac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}\right)$$

$$= \arctan\left(\tan\left(\frac{\pi}{7}\right)\right)$$

$$= \frac{\pi}{7}$$

biii

For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $w = \text{cis}\left(\frac{2\pi}{7}\right)$ is a root, $w^* = \text{cis}\left(-\frac{2\pi}{7}\right)$ is also a root.

Therefore $(z-w)(z-w^*)$ must be a factor of the polynomial $z^7 - 1$

$$(z-w)(z-w^*) = z^2 - 2\text{Re}\,w + |w|^2$$

$$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$$

By the same argument,

$\left(z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1\right)$ and $\left(z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1\right)$ must also be factors.

46 **ai**

Using binomial theorem to expand

$$(\text{cis}\,\theta)^5 = \cos^5\theta + 5\text{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\text{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \text{i}\sin^5\theta$$

aii

By De Moivre's theorem, $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$ so $(\text{cis}\,\theta)^5 = \text{cis}\,5\theta$

Then $\text{Im}\,((\text{cis}\,\theta)^5) = \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

aiii

$$\text{Re}\,((\text{cis}\,\theta)^5) = \cos 5\theta$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

b

$z^5 = r^5\,\text{cis}\,5\alpha = 1 = \text{cis}(360n°)$ for $n \in \mathbb{Z}$

$r^5 = 1$ so $r = 1$

$5\alpha = 360n°$ and $\alpha > 0$ is the minimum possible value, so $\alpha = 72°$

c

Since $\text{Im}\,(z^5) = \text{Im}\,(1) = 0$, by part **aii**,

$5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha = 0$

Since $\sin\alpha \neq 0$,

$5\cos^4\alpha - 10\cos^2\alpha\sin^2\alpha + \sin^4\alpha = 0$

$5(1-\sin^2\alpha)^2 - 10(1-\sin^2\alpha)\sin^2\alpha + \sin^4\alpha = 0$

$16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$

d

The result in part **c** is a quadratic in $\sin^2\alpha$

Using the quadratic formula:

$$\sin^2\alpha = \frac{20 \pm \sqrt{(-20)^2 - 4\times 16\times 5}}{2\times 16}$$
$$= \frac{10 \pm 2\sqrt{5}}{16}$$

Then $\sin\alpha = \sqrt{\dfrac{10+2\sqrt{5}}{16}} = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

$a = 10, b = 2, c = 5, d = 4$

47 **a**

Using the binomial theorem:

$$(\text{cis}\,\theta)^3 = \cos^3\theta + 3\text{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \text{i}\sin^3\theta$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + \text{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$$

b

By De Moivre's theorem, $(\text{cis}\,\theta)^3 = \text{cis}\,3\theta$

So $\text{Re}\,(\text{cis}\,3\theta) = \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$
$$= 4\cos^3\theta - 3\cos\theta$$

c

By similar reasoning and the binomial expansion,

$$\cos 5\theta = \text{Re}\,((\text{cis}\,\theta)^5) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

d

$$\cos 5\theta + \cos 3\theta + \cos\theta = 16\cos^5\theta - 16\cos^3\theta + 3\cos\theta$$
$$= \cos\theta(4\cos^2\theta - 1)(4\cos^2\theta - 3)$$

So $\cos\theta = 0$ or $\pm\dfrac{1}{2}$ or $\pm\dfrac{\sqrt{3}}{2}$

$\theta = \pm\dfrac{\pi}{2}, \pm\dfrac{\pi}{3}, \pm\dfrac{\pi}{6}$

e

$$\cos 5\theta = 0 = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$
$$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$$

So $\cos\theta = 0$ or $\cos^2\theta = \dfrac{20 \pm \sqrt{(-20)^2 - 4\times 16\times 5}}{2\times 16} = \dfrac{5\pm\sqrt{5}}{8}$

But the smallest possible positive solution to $\cos 5\theta = 0$ is $5\theta = \frac{\pi}{2}$ so $\theta = \frac{\pi}{10}$

This must correspond to the largest possible value of $\cos\theta$

So $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$

The other solutions to $\cos 5\theta = 0$ are $5\theta = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ or $\frac{9\pi}{2}$

So $\theta = \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$

The cosine of each of these, given the decreasing value of the cosine curve between 0 and π, must be the remaining four roots of the quintic in decreasing order:

$\cos\left(\frac{3\pi}{10}\right) = \sqrt{\frac{5-\sqrt{5}}{8}}$

$\cos\left(\frac{\pi}{2}\right) = 0$

$\cos\left(\frac{7\pi}{10}\right) = -\sqrt{\frac{5-\sqrt{5}}{8}}$

$\cos\left(\frac{9\pi}{10}\right) = -\sqrt{\frac{5+\sqrt{5}}{8}}$

5 Mathematical proof

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 5A

1

Proposition: $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

Base case $n = 1$: $1^3 = 1 = \frac{1^2(2^2)}{4}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$

Working towards: $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{(by assumption)} \\ &= \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4}[k^2 + 4k + 4] \\ &= \frac{(k+1)^2}{4}(k+2)^2 \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

2

Proposition: $1 \times 3 + 2 \times 4 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Base case $n = 1$: $1 \times 3 = 3 = \frac{1(2)(9)}{6}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $1 \times 3 + 2 \times 4 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$

Working towards: $1 \times 3 + 2 \times 4 + \cdots + k(k+2) + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}$

$1 \times 3 + 2 \times 4 + \cdots + k(k+2) + (k+1)(k+3)$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

(by assumption)

$$\begin{aligned} &= \frac{k+1}{6}[k(2k+7) + 6(k+3)] \\ &= \frac{k+1}{6}[2k^2 + 13k + 18] \end{aligned}$$

$$= \frac{(k+1)}{6}(k+2)(2k+9)$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

3

Proposition: $\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$

Base case $n = 1$: $1^2 \times (1+1) = 2 = \frac{1(2)(3)(4)}{12}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12}$

Working towards: $\sum_{r=1}^{k+1} r^2(r+1) = \frac{(k+1)(k+2)(k+3)(3k+4)}{12}$

$$\sum_{r=1}^{k+1} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2) \text{(by assumption)}$$

$$= \frac{(k+1)(k+2)}{12}[k(3k+1) + 12(k+1)]$$

$$= \frac{(k+1)(k+2)}{12}[3k^2 + 13k + 12]$$

$$= \frac{(k+1)(k+2)}{12}(k+3)(3k+4)$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

4

Proposition: $\sum_{r=1}^{n} 2 \times 3^{r-1} = 3^n - 1$

Base case $n = 1$: $2 \times 3^0 = 2 = 3^1 - 1$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} 2 \times 3^{r-1} = 3^k - 1$

Working towards: $\sum_{r=1}^{k+1} 2 \times 3^{r-1} = 3^{k+1} - 1$

$$\sum_{r=1}^{k+1} 2 \times 3^{r-1} = \sum_{r=1}^{k} 2 \times 3^{r-1} + 2 \times 3^{k+1-1}$$

$$= 3^k - 1 + 2 \times 3^k \text{(by assumption)}$$

$$= 3 \times 3^k - 1$$

$$= 3^{k+1} - 1$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

5

Proposition: $\sum_{r=1}^{n} \frac{1}{(2r-1)\times(2r+1)} = \frac{n}{2n+1}$

Base case $n = 1$: $\frac{1}{1\times3} = \frac{1}{3} = \frac{1}{2\times1+1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} \frac{1}{(2r-1)\times(2r+1)} = \frac{k}{2k+1}$

Working towards: $\sum_{r=1}^{k+1} \frac{1}{(2r-1)\times(2r+1)} = \frac{k+1}{2k+3}$

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{1}{(2r-1)\times(2r+1)} &= \sum_{r=1}^{k} \frac{1}{(2r-1)\times(2r+1)} + \frac{1}{(2k+1)\times(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)\times(2k+3)} \text{ (by assumption)} \\ &= \frac{1}{2k+1}\left[k + \frac{1}{2k+3}\right] \\ &= \frac{1}{2k+1} \times \frac{[k(2k+3)+1]}{2k+3} \\ &= \frac{1}{2k+1} \times \frac{[2k^2+3k+1]}{2k+3} \\ &= \frac{1}{2k+1} \times \frac{(2k+1)(k+1)}{2k+3} \\ &= \frac{k+1}{2k+3}\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

6

Proposition: $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

Base case $n = 1$: $\frac{1}{1\times2} = \frac{1}{2} = \frac{1}{1+1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

Working towards: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{1}{r(r+1)} &= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \text{ (by assumption)} \\ &= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right] \\ &= \frac{1}{k+1} \times \frac{[k(k+2)+1]}{k+2} \\ &= \frac{1}{k+1} \times \frac{[k^2+2k+1]}{k+2}\end{aligned}$$

$$= \frac{1}{k+1} \times \frac{(k+1)^2}{k+2}$$
$$= \frac{k+1}{k+2}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

7

Proposition: $\sum_{r=1}^{n} 3r(r-1) = n(n^2 - 1)$

Base case $n = 1$: $3 \times 1 \times (1 - 1) = 0 = 1(1^2 - 1)$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} 3r(r-1) = k(k^2 + 1)$

Working towards:

$$\sum_{r=1}^{k+1} 3r(r-1) = (k+1)((k+1)^2 - 1) = (k+1)(k^2 + 2k) = k(k+1)(k+2)$$

$$\sum_{r=1}^{k+1} 3r(r-1) = \sum_{r=1}^{k} 3r(r-1) + 3(k+1)(k+1-1)$$
$$= k(k^2 + 1) + 3(k+1)k \text{(by assumption)}$$
$$= k[k^2 + 1 + 3(k+1)]$$
$$= k[k^2 + 3k + 2]$$
$$= k(k+1)(k+2)$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

8

Proposition: $5^n - 1 = 4m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $5^0 - 1 = 0 = 4 \times 0$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $5^k - 1 = 4A$ for some $A \in \mathbb{Z}$

Working towards: $5^{k+1} - 1 = 4B$ for some $B \in \mathbb{Z}$

$$5^{k+1} - 1 = 5(5^k) - 1$$
$$= 5(5^k - 1) + 4$$
$$= 5(4A) + 4 \text{(by assumption)}$$
$$= 4(5A + 1)$$
$$= 4B \text{(where } B = 5A + 1 \in \mathbb{Z})$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

9

Proposition: $4^n - 1 = 3m$ for some $m \in \mathbb{Z}$
Base case $n = 1$: $4^1 - 1 = 3 = 3 \times 1$ so the proposition is true for $n = 1$
Inductive step: Assume the proposition is true for integer $n = k \geq 1$
So $4^k - 1 = 3A$ for some $A \in \mathbb{Z}$
Working towards: $4^{k+1} - 1 = 3B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}4^{k+1} - 1 &= 4(4^k) - 1\\ &= 4(4^k - 1) + 3\\ &= 4(3A) + 3 \text{(by assumption)}\\ &= 3(4A + 1)\\ &= 3B \text{(where } B = 4A + 1 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$
Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

10

Proposition: $7^n - 3^n = 4m$ for some $m \in \mathbb{Z}$
Base case $n = 0$: $7^0 - 3^0 = 0 = 4 \times 0$ so the proposition is true for $n = 0$
Inductive step: Assume the proposition is true for integer $n = k \geq 1$
So $7^k - 3^k = 4A$ for some $A \in \mathbb{Z}$
Working towards: $7^{k+1} - 3^{k+1} = 4B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}7^{k+1} - 3^{k+1} &= 7(7^k) - 3(3^k)\\ &= 7(7^k - 3^k) + 4 \times 3^k\\ &= 7(4A) + 4 \times 3^k \text{(by assumption)}\\ &= 4(7A + 3^k)\\ &= 4B \text{(where } B = 7A + 3^k \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$
Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11

Proposition: $30^n - 6^n = 12m$ for some $m \in \mathbb{Z}$
Base case $n = 0$: $30^0 - 6^0 = 0 = 12 \times 0$ so the proposition is true for $n = 0$
Inductive step: Assume the proposition is true for integer $n = k \geq 1$
So $30^k - 6^k = 12A$ for some $A \in \mathbb{Z}$
Working towards: $30^{k+1} - 6^{k+1} = 12B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}30^{k+1} - 6^{k+1} &= 30(30^k) - 6(6^k)\\ &= 30(30^k - 6^k) + 24 \times 6^k\\ &= 30(12A) + 24 \times 6^k \text{(by assumption)}\\ &= 12(30A + 2 \times 6^k)\\ &= 12B \text{(where } B = 30A + 2 \times 6^k \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$
Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 0$ by the principle of mathematical induction.

12

Proof uses the fact that the product of two consecutive numbers must be even.
That is, for any integer k, because either k or $k+1$ must be even it follows that
$k(k+1) = 2C$ for some integer C $(*)$

Proposition: $n^3 - n = 6m$ for some $m \in \mathbb{Z}$

Base case $n = 1$: $1^3 - 1 = 0 = 6 \times 0$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $k^3 - k = 6A$ for some $A \in \mathbb{Z}$

Working towards: $(k+1)^3 - (k+1) = 6B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 - k + 3k^2 + 3k \\ &= 6A + 3k(k+1)(\text{by assumption}) \\ &= 6A + 3 \times 2C(\text{by } *, \text{for some integer } C) \\ &= 6(A+C)(\text{where } B = A + C \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 1$ by the principle of mathematical induction.

13

Proof uses the fact that the product of two consecutive numbers must be even.
That is, for any integer k, because either k or $k+1$ must be even it follows that
$k(k+1) = 2C$ for some integer C $(*)$

Proposition: $n(n^2 + 5) = 6m$ for some $m \in \mathbb{Z}$

Base case $n = 1$: $1(1^2 + 5) = 6 = 6 \times 1$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $k(k^2 + 5) = 6A$ for some $A \in \mathbb{Z}$

Working towards: $(k+1)((k+1)^2 + 5) = 6B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}(k+1)((k+1)^2 + 5) &= (k+1)(k^2 + 2k + 6) \\ &= k^3 + 3k^2 + 8k + 6 \\ &= k^3 + 5k + (3k^2 + 3k + 6) \\ &= k^3 + 5k + 3k(k+1) + 6 \\ &= 6A + 3k(k+1) + 6(\text{by assumption}) \\ &= 6A + 3 \times 2C + 6(\text{by } *, \text{for some integer } C) \\ &= 6(A + C + 1)(\text{where } B = A + C + 1 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 1$ by the principle of mathematical induction.

14

Main proof uses the fact that for any integer $k \geq 1$, $4^k - 1 = 3m$ for some integer m $(*)$

(See worked solution for Q9 above for proof)

Proposition: $7^n - 4^n - 3n = 9m$ for some $m \in \mathbb{Z}$

Base case $n = 1$: $7^1 - 4^1 - 3 \times 1 = 7 - 4 - 3 = 0 = 0 \times 9$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $7^k - 4^k - 3k = 9A$ for some $A \in \mathbb{Z}$

Working towards: $7^{k+1} - 4^{k+1} - 3(k+1) = 9B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}7^{k+1} - 4^{k+1} - 3(k+1) &= 7(7^k) - 4(4^k) - 3(k+1)\\ &= 7(7^k - 4^k - 3k) + 3 \times 4^k + 18k - 3\\ &= 7(7^k - 4^k - 3k) + 18k + 3(4^k - 1)\\ &= 7(9A) + 18k + 3(4^k - 1)(\text{by assumption})\\ &= 7(9A) + 18k + 3(3m)(\text{by } *, \text{for some integer } C)\\ &= 9(7A + 2k + m)\ (\text{where } B = 7A + 2k + m \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 1$ by the principle of mathematical induction.

15

Proposition: $3^{2n+2} - 8n - 9 = 64m$ for some $m \in \mathbb{Z}$

Base case $n = 1$: $3^{2\times1+2} - 8 \times 1 - 9 = 81 - 8 - 9 = 64 = 1 \times 64$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $3^{2k+2} - 8k - 9 = 64A$ for some $A \in \mathbb{Z}$

Working towards: $3^{2k+4} - 8(k+1) - 9 = 64B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}3^{2k+4} - 8(k+1) - 9 &= 9(3^{2k+2}) - 8k - 17\\ &= 9(3^{2k+2} - 8k - 9) + 64k + 64\\ &= 9(64A) + 64(k+1)(\text{by assumption})\\ &= 64(9A + k + 1)\ (\text{where } B = 9A + k + 1 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

16

$u_1 = 7, u_{n+1} = 2u_n + 3$ for $n \geq 1$

Proposition: $u_n = 5 \times 2^n - 3$

Base case $n = 1$: $7 = 5 \times 2^1 - 3$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $u_k = 5 \times 2^k - 3$

Working towards: $u_{k+1} = 5 \times 2^{k+1} - 3$

$$\begin{aligned}u_{k+1} &= 2u_k + 3(\text{recurrence relation})\\ &= 2(5 \times 2^k - 3) + 3(\text{by assumption})\\ &= 5 \times 2^{k+1} - 6 + 3\\ &= 5 \times 2^{k+1} - 3\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

17

$u_1 = 3, u_{n+1} = 5u_n - 8$ for $n \geq 1$

Proposition: $u_n = 5^{n-1} + 2$

Base case $n = 1$: $3 = 5^0 + 2$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $u_k = 5^{k-1} + 2$

Working towards: $u_{k+1} = 5^k + 2$

$$\begin{aligned} u_{k+1} &= 5u_k - 8 \text{(recurrence relation)} \\ &= 5(5^{k-1} + 2) - 8 \text{(by assumption)} \\ &= 5^k + 10 - 8 \\ &= 5^k + 2 \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

18

$u_1 = 1, u_{n+1} = 3u_n + 1$ for $n \geq 1$

Proposition: $u_n = \frac{1}{2}(3^n - 1)$

Base case $n = 1$: $1 = \frac{1}{2}(3^1 - 1)$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $u_k = \frac{1}{2}(3^k - 1)$

Working towards: $u_{k+1} = \frac{1}{2}(3^{k+1} - 1)$

$$\begin{aligned} u_{k+1} &= 3u_k + 1 \text{(recurrence relation)} \\ &= 3\left(\frac{1}{2}(3^k - 1)\right) + 1 \text{(by assumption)} \\ &= \frac{1}{2} \times 3^{k+1} - \frac{3}{2} + 1 \\ &= \frac{1}{2}(3^{k+1} - 1) \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

19

$y = (1 - x)^{-1}$

Proposition: $y^{(n)} = n!\,(1 - x)^{-n-1}$

Base case $n = 0$: $y^{(0)} = y = 0!\,(1 - x)^{0-1}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $y^{(k)} = k!\,(1 - x)^{-k-1}$

Working towards: $y^{(k+1)} = (k+1)!\,(1-x)^{-k-2}$

$$\begin{aligned} y^{(k+1)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(k)}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(k!\,(1-x)^{-k-1}\right) \text{(by assumption)} \\ &= (-1)(-k-1)k!\,(1-x)^{-k-2} \text{(by chain rule)} \\ &= (k+1)k!\,(1-x)^{-k-2} \\ &= (k+1)!\,(1-x)^{-k-2} \end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

20

$y = (1-3x)^{-1}$

Proposition: $y^{(n)} = 3^n n!\,(1-3x)^{-n-1}$

Base case $n = 0$: $y^{(0)} = y = 3^0 0!\,(1-3x)^{0-1}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $y^{(k)} = 3^k k!\,(1-3x)^{-k-1}$

Working towards: $y^{(k+1)} = 3^{k+1}(k+1)!\,(1-3x)^{-k-2}$

$$\begin{aligned} y^{(k+1)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(k)}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3^k k!\,(1-3x)^{-k-1}\right) \text{(by assumption)} \\ &= (-3)(-k-1)3^k k!\,(1-3x)^{-k-2} \text{(by chain rule)} \\ &= (k+1)3^{k+1}k!\,(1-3x)^{-k-2} \\ &= 3^{k+1}(k+1)!\,(1-3x)^{-k-2} \end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

21

$y = x\mathrm{e}^{2x}$

Proposition: $y^{(n)} = (2^n x + n2^{n-1})\mathrm{e}^{2x}$

Base case $n = 0$: $y^{(0)} = y = (2^0 x + 0 \times 2^{-1})e^{2x}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $y^{(k)} = (2^k x + k2^{k-1})\mathrm{e}^{2x}$

Working towards: $y^{(k+1)} = (2^{k+1}x + (k+1)2^k)\mathrm{e}^{2x}$

$$\begin{aligned} y^{(k+1)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(k)}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left((2^k x + k2^{k-1})\mathrm{e}^{2x}\right) \text{(by assumption)} \\ &= 2^k\mathrm{e}^{2x} + 2(2^k x + k2^{k-1})\mathrm{e}^{2x} \text{(by product rule)} \\ &= (2^k + 2^{k+1}x + k2^k)\mathrm{e}^{2x} \\ &= (2^{k+1}x + (k+1)2^k)\mathrm{e}^{2x} \end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

22

$y = x \sin x$

Proposition: $y^{(2n)} = (-1)^n (x \sin x - 2n \cos x)$

Base case $n = 0$: $y^{(0)} = y = (-1)^0 (x \sin x + 0 \cos x)$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $y^{(2k)} = (-1)^k (x \sin x - 2k \cos x)$

Working towards: $y^{(2k+2)} = (-1)^{k+1}(x \sin x - 2(k+1) \cos x)$

$$
\begin{aligned}
y^{(2k+2)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(2k)}\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}((-1)^k (x \sin x - 2k \cos x)\right) \text{(by assumption)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}((-1)^k (\sin x + x \cos x + 2k \sin x) \\
&= (-1)^k (\cos x + \cos x - x \sin x + 2k \cos x) \\
&= (-1)^k \left((2k+2) \cos x - x \sin x\right) \\
&= (-1)^{\mathrm{k}+1} (x \sin x - 2(k+1) \cos x)
\end{aligned}
$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

23

$y = x^2 \mathrm{e}^x$

Then $y' = (2x + x^2)\mathrm{e}^x$

$y'' = (2 + 2x + 2x + x^2)\mathrm{e}^x = (2 + 4x + x^2)\mathrm{e}^x$

Proposition: $y^{(n)} = (x^2 + 2nx + n(n-1))e^x$ for $n \geq 2$

Base case $n = 2$: $y^{(2)} = y'' = \left(x^2 + 2(2)x + 2(1)\right)\mathrm{e}^x$ so the proposition is true for $n = 2$

Inductive step: Assume the proposition is true for integer $n = k \geq 2$

So $y^{(k)} = \left(x^2 + 2kx + k(k-1)\right)\mathrm{e}^x$

Working towards: $y^{(k+1)} = (x^2 + 2(k+1)x + (k+1)k)\mathrm{e}^x$

$$
\begin{aligned}
y^{(k+1)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(k)}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2 + 2kx + k(k-1)\right)\mathrm{e}^x\right) \text{(by assumption)} \\
&= \left(x^2 + 2k + 2x + 2kx + k(k-1)\right)\mathrm{e}^x \\
&= \left(x^2 + 2(k+1)x + k(2 + k - 1)\right)\mathrm{e}^x \\
&= \left(x^2 + 2(k+1)x + k(k+1)\right)\mathrm{e}^x
\end{aligned}
$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 2$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 2$ by the principle of mathematical induction.

24 **a**

For some real values u, v, x, y, $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$

Then $z^* = x - \mathrm{i}y$ and $w^* = u - \mathrm{i}v$

$(zw)^* = \big((x + \mathrm{i}y)(u + \mathrm{i}v)\big)^*$

$= \big(xu - yv + \mathrm{i}(xv + yu)\big)^*$

$= xu - yv - \mathrm{i}(xv + yu)$

$z^*w^* = (x - \mathrm{i}y)(u - \mathrm{i}v)$

$= xu - yv - \mathrm{i}(xv + yu)$

So $(zw)^* \equiv z^*w^*$

b

Proposition: $(z^n)^* = (z^*)^n$ for all positive integers n.

Base case $n = 1$: $(z^*) = z^* = (z^1)^*$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $(z^k)^* = (z^*)^k$

Working towards: $(z^{k+1})^* = (z^*)^{k+1}$

$(z^*)^{k+1} = (z^*)^k \times z^*$

$= (z^k)^* \times z^*$(by assumption)

$= (z^k \times z)^*$(by part **a**)

$= (z^{k+1})^*$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

25

$u_1 = 5$, $u_2 = 13$, $u_{n+2} = 5u_{n+1} - 6u_n$ for $n \geq 1$

Proposition: $u_n = 2^n + 3^n$

Base case $n = 1$: $5 = 2^1 + 3^1$ so the proposition is true for $n = 1$

Base case $n = 2$: $13 = 2^2 + 3^2$ so the proposition is true for $n = 2$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$ and for $n = k + 1$

So $u_k = 2^k + 3^k$ and $u_{k+1} = 2^{k+1} + 3^{k+1}$

Working towards: $u_{k+2} = 2^{k+2} + 3^{k+2}$

$u_{k+2} = 5u_{k+1} - 6u_k$(recurrence relation)

$= 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$(by assumption)

$= 2^k(10 - 6) + 3^k(15 - 6)$

$= 4 \times 2^k + 9 \times 3^k$

$= 2^{k+2} + 3^{k+2}$

So the proposition is true for $n = k + 2$

Conclusion:
The proposition is true for $n = 1$ and $n = 2$, and, if true for $n = k$ and $n = k + 1$, it is also true for $n = k + 2$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

26

$u_1 = 3, u_2 = 36, u_{n+2} = 6u_{n+1} - 9u_n$ for $n \geq 1$

Proposition: $u_n = (3n-2)3^n$

Base case $n = 1$: $3 = (3 \times 1 - 2) \times 3^1$ so the proposition is true for $n = 1$

Base case $n = 2$: $36 = (3 \times 2 - 2) \times 3^2$ so the proposition is true for $n = 2$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$ and for $n = k+1$

So $u_k = (3k-2)3^k$ and $u_{k+1} = (3k+1)3^{k+1}$

Working towards: $u_{k+2} = (3k+4)3^{k+2}$

$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 9u_k \text{(recurrence relation)} \\ &= 6\left((3k+1)3^{k+1}\right) - 9\left((3k-2)3^k\right) \text{(by assumption)} \\ &= [18(3k+1) - 9(3k-2)]3^k \\ &= 9[6k+2-3k+2]3^k \\ &= [3k+4]9 \times 3^k \\ &= (3k+4)3^{k+2} \end{aligned}$$

So the proposition is true for $n = k+2$

Conclusion:

The proposition is true for $n = 1$ and $n = 2$, and, if true for $n = k$ and $n = k+1$, it is also true for $n = k+2$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

27

Proposition: $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$

Base case $n = 1$: $1 \times 1! = 1 = (1+1)! - 1$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} r \times r! = (k+1)! - 1$

Working towards: $\sum_{r=1}^{k+1} r \times r! = (k+2)! - 1$

$$\begin{aligned} \sum_{r=1}^{k+1} r \times r! &= \sum_{r=1}^{k} r \times r! + (k+1) \times (k+1)! \\ &= (k+1)! - 1 + (k+1) \times (k+1)! \text{ (by assumption)} \\ &= (k+1)!\,[1 + k + 1] - 1 \\ &= (k+1)! \times (k+2) - 1 \\ &= (k+2)! - 1 \end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

28

Proposition: $\sum_{r=1}^{n} (-1)^{r-1} r^2 = (-1)^{n-1} \frac{n(n+1)}{2}$

Base case $n = 1$: $(-1)^0 \times 1^2 = 1 = (-1)^0 \times \frac{1 \times (1+1)}{2}$ so the proposition is true for $n =$ 1

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} (-1)^{r-1} r^2 = (-1)^{k-1} \frac{k(k+1)}{2}$

Working towards: $\sum_{r=1}^{k+1} (-1)^{r-1} r^2 = (-1)^k \frac{(k+1)(k+2)}{2}$

$$\sum_{r=1}^{k+1}(-1)^{r-1}r^2 = \sum_{r=1}^{k}(-1)^{r-1}r^2 + (-1)^k(k+1)^2$$
$$= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k(k+1)^2 \text{(by assumption)}$$
$$= (-1)^k\frac{(k+1)}{2}[-k+2(k+1)]$$
$$= \frac{(-1)^k(k+1)}{2}(k+2)$$
$$= (-1)^k\frac{(k+1)(k+2)}{2}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

29

Tip: This could be proved using a difference between two triangle numbers. In the spirit of the chapter, the proof by induction is given below. Remember that in an exam, if the question does not specify a method, you may choose any valid method as long as you ensure that you make no assumptions not permitted within the spirit of the question – avoiding circular arguments, in particular!

Proposition: $\sum_{r=1}^{n}(n+r) = \frac{1}{2}n(3n+1)$

Base case $n = 1$: $1+1 = \frac{1}{2}\times 1 \times (3+1)$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k}(k+r) = \frac{1}{2}k(3k+1)$

Working towards: $\sum_{r=1}^{k+1}(k+1+r) = \frac{1}{2}(k+1)(3k+4)$

$$\sum_{r=1}^{k+1}(k+1+r) = \sum_{r=1}^{k}(k+1+r) + (k+1+k+1)$$
$$= \sum_{r=1}^{k}(k+r) + \sum_{r=1}^{k} 1 + 2k + 2$$
$$= \sum_{r=1}^{k}(k+r) + k + 2k + 2$$
$$= \frac{1}{2}k(3k+1) + 3k + 2 \text{(by assumption)}$$
$$= \frac{1}{2}[3k^2 + k + 6k + 4]$$
$$= \frac{1}{2}(3k^2 + 7k + 4)$$
$$= \frac{1}{2}(k+1)(3k+4)$$

So the proposition is true for $n = k+1$

Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

30

Tip: The question uses k as the index variable in the sum; either use a different index variable or a different variable for the inductive argument – make sure you don't use the same letter in different contexts within the same question.

Proposition:

$$\sum_{k=1}^{n} k\,2^k = (n-1)2^{n+1} + 2$$

Base case $n = 1$:

$1 \times 2^1 = (1-1) \times 2^2 + 2$ so the proposition is true for $n = 1$

Inductive step:

Assume the proposition is true for integer $n = m \geq 1$

So $\displaystyle\sum_{k=1}^{m} k\,2^k = (m-1)2^{m+1} + 2$

Working towards: $\sum_{k=1}^{m+1} k2^k = m2^{m+2} + 2$

$$\begin{aligned}\sum_{k=1}^{m+1} k2^k &= \sum_{k=1}^{m} k2^k + (m+1)2^{m+1}\\ &= (m-1)2^{m+1} + 2 + (m+1)2^{m+1} \text{(by assumption)}\\ &= 2^{m+1}(m-1+m+1) + 2\\ &= 2^{m+1} \times 2m + 2\\ &= m2^{m+2} + 2\end{aligned}$$

So the proposition is true for $n = m + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = m$, it is also true for $n = m + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

31

Proposition: For any $n \in \mathbb{Z}$, $n^3 + (n+1)^3 + (n+2)^3 = 9m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $0^3 + 1^3 + 2^3 = 9 = 9 \times 1$ so the proposition is true for $n = 0$

Inductive step (positive direction): Assume the proposition is true for integer $n = k \geq 0$

So $k^3 + (k+1)^3 + (k+2)^3 = 9A$ for some $A \in \mathbb{Z}$

Working towards: $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k+1)^3 + (k+2)^3 + [k^3 + 9k^2 + 27k + 27]\\ &= k^3 + (k+1)^3 + (k+2)^3 + 9(k^2 + 3k + 3)\\ &= 9A + 9(k^2 + 3k + 3) \text{(by assumption)}\\ &= 9B \text{(where } B = A + k^2 + 3k + 3 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Inductive step (negative direction): Assume the proposition is true for integer

$$n = k \le 0$$

So $k^3 + (k+1)^3 + (k+2)^3 = 9A$ for some $A \in \mathbb{Z}$

Working towards: $(k-1)^3 + k^3 + (k+1)^3 = 9C$ for some $C \in \mathbb{Z}$

$$\begin{aligned}(k-1)^3 + k^3 + (k+1)^3 &= [k^3 - 3k^2 + 3k - 1] + k^3 + (k+1)^3 \\ &= [k^3 + 6k^2 + 12k + 8 - 9k^2 - 9k - 9] + k^3 + (k+1)^3 \\ &= (k+2)^3 - 9(k^2 + k + 1) + k^3 + (k+1)^3 \\ &= 9A - 9(k^2 + k + 1)(\text{by assumption}) \\ &= 9C(\text{where } C = A - k^2 - k - 1 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k - 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k \pm 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}$ by the principle of mathematical induction.

32

Proposition: $2 \times 6 \times 10 \times \ldots \times (4n - 2) = \frac{(2n)!}{n!}$ for some $n \ge 1$

Base case $n = 1$: $2 = \frac{(2\times1)!}{1!} = \frac{2}{1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \ge 1$

So $2 \times 6 \times 10 \times \ldots \times (4k - 2) = \frac{(2k)!}{k!}$

Working towards: $2 \times 6 \times 10 \times \ldots \times (4k-2) \times (4k+2) = \frac{(2k+2)!}{(k+1)!}$

$$\begin{aligned}2 \times 6 \times 10 \times \ldots \times (4k-2) \times (4k+2) &= [2 \times 6 \times 10 \times \ldots \times (4k-2)] \times (4k+2) \\ &= \frac{2k!}{k!} \times (4k+2)(\text{by assumption}) \\ &= \frac{2k!}{k!} \times 2(2k+1) \\ &= \frac{2k!}{k!} \times \frac{2(2k+1)(2k+2)}{2k+2} \\ &= \frac{2k!}{k!} \times \frac{(2k+1)(2k+2)}{k+1} \\ &= \frac{(2k+2)!}{(k+1)!}\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \ge 1$ by the principle of mathematical induction.

33

$u_1 = 1, u_{n+1} = \dfrac{u_n}{u_n + 1}$ for $n \ge 1$

Proposition: $u_n = \frac{1}{n}$

Base case $n = 1$: $1 = \frac{1}{1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \ge 1$

So $u_k = \frac{1}{k}$

Working towards: $u_{k+1} = \frac{1}{k+1}$

$$u_{k+1} = \frac{u_k}{u_k + 1} \text{(recurrence relation)}$$
$$= \frac{\frac{1}{k}}{\frac{1}{k} + 1} \text{(by assumption)}$$
$$= \frac{\left(\frac{1}{k}\right)}{\frac{1}{k} + 1} \times \frac{k}{k}$$
$$= \frac{1}{k+1}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

34

Proposition: $2^n > 1 + n$ for all $n > 1$

Base case $n = 2$: $2^2 = 4 > 1 + 2$ so the proposition is true for $n = 2$

Inductive step: Assume the proposition is true for integer $n = k \geq 2$

So $2^k > 1 + k$

Working towards: $2^{k+1} > 2 + k$

$$2^{k+1} = 2 \times 2^k$$
$$> 2 \times (1 + k) \text{(by assumption)}$$
$$> 2 + 2k$$
$$> (2 + k) + k > 2 + k \text{ (since } k > 0)$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 2$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 1$ by the principle of mathematical induction.

35

The proof uses the fact that for $p \geq 3$, $p^2 > 2$ $\qquad (*)$

Proposition: $2^n > n^2$ for all $n \geq 4$

Base case $n = 4$: $2^4 = 16 = 4^2$ so the proposition is true for $n = 4$

Inductive step: Assume the proposition is true for integer $n = k \geq 4$

So $2^k \geq k^2$

Working towards: $2^{k+1} > (k + 1)^2$

$$2^{k+1} = 2 \times 2^k$$
$$\geq 2 \times (k^2) \text{(by assumption)}$$
$$\geq k^2 + 2k + 1 + k^2 - 2k - 1$$
$$\geq (k + 1)^2 + k^2 - 2k + 1 - 2$$
$$\geq (k + 1)^2 + (k - 1)^2 - 2$$
$$> (k + 1)^2 \left(\text{since } (k - 1)^2 > 2 \text{ by } (*)\right)$$

So the proposition is true for $n = k + 1$
Conclusion:
The proposition is true for $n = 4$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 4$ by the principle of mathematical induction.

Exercise 5B

Tip

Several questions in this exercise require assumption or proof of a basic rule of prime factorisation. These can generally be assumed (unless specified otherwise) but in the worked solutions below they are often explicitly proved. When sitting an examination, students may choose, depending on circumstance and the credit available for a question, whether these should be proved or merely asserted. Asserting is generally acceptable unless it leads to a circular argument.

1

Proposition: For integer n, if n^2 is even then n must be even
Assume the contrary: Suppose that n^2 is even and n is odd.
Then $n = 2k + 1$ for some integer k
So $n^2 = (2k + 1)(2k + 1)$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2m + 1$ for some integer m
So n^2 is odd, which contradicts the assumption.
Conclusion:
The assumption that if n^2 is even n could be odd is shown to lead to a contradiction.
Therefore, if n^2 is even, n must also be even.

2

Proposition: For integers a and b, if ab is even then a must be even or b must be even
Assume the contrary: Suppose that ab is even and both a and b are odd.
Then $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n.
So $ab = (2m + 1)(2n + 1)$
$= 4mn + 2m + 2n + 1$
$= 2(2mn + m + n) + 1$
$= 2k + 1$ for some integer k
So ab is odd, which contradicts the assumption.
Conclusion:
The assumption that if ab is even both a and b could be odd is shown to lead to a contradiction.
Therefore, if ab is even, at least one of a and b must also be even.

3

The proof uses the fact $(*)$ that if an integer k has square k^2 which is a multiple of 5 then k must also be a multiple of 5.
Proving $(*)$ first, by contradiction:
Proposition: For integer k, if $k^2 = 5m$ for some integer m then $k = 5n$ for some integer n

Assume the contrary: Suppose that $k^2 = 5n$ and $k = 5n + r$ where r is an integer $1 \leq r \leq 4$.
So $k^2 = (5n + r)(5n + r)$
$= 25n^2 + 10nr + r^2$
$= 5(5n^2 + 2nr) + r^2$
$r^2 = 1, 4, 9$ or 16, none of which are multiples of 5
So k^2 is not a multiple of 5, which contradicts the assumption.
Conclusion:
The assumption that, for an integer k, k^2 can be a multiple of 5 while k is not a multiple of 5 is shown to lead to a contradiction.
Therefore, if k^2 is a multiple of 5, k must also be a multiple of 5.

Now to the question asked:
Proposition: $\sqrt{5}$ is irrational
Assume the contrary: Suppose that $\sqrt{5}$ can be expressed as $\frac{p}{q}$ for some integers p and q which have no common factors greater than 1 (i.e. the fraction is expressed in simplified form).
Squaring both sides:
$5 = \frac{p^2}{q^2}$
$p^2 = 5q^2$
Then p^2 is a multiple of 5 and, by $(*)$, p must also be a multiple of 5
$p = 5a$ for some integer a.
$(5a)^2 = 5q^2$
$25a^2 = 5q^2$
$q^2 = 5a^2$
Then q^2 is a multiple of 5 and, by $(*)$, q must also be a multiple of 5.
This contradicts the assumption that p and q have no common factors greater than 1.
Conclusion:
The assumption that $\sqrt{5} = \frac{p}{q}$ for some integers p and q with $\gcd(p, q) = 1$ is shown to lead to a contradiction.
Therefore, $\sqrt{5}$ cannot be expressed as a ratio of integers in simplest terms.
Therefore, $\sqrt{5}$ is irrational.

4
The proof uses the fact $(*)$ that if a power of an integer is even, that integer must also be even.
$(*)$ is assumed without proof here (proof is similar to that given at the start of question 3)

Proposition: $\sqrt[3]{2}$ is irrational
Assume the contrary: Suppose that $\sqrt[3]{2}$ can be expressed as $\frac{p}{q}$ for some integers p and q which have no common factors greater than 1 (i.e. the fraction is expressed in simplified form).

Cubing both sides:

$2 = \frac{p^3}{q^3}$

$p^3 = 2q^3$

Then p^3 is a multiple of 2 and, by $(*)$, p must also be a multiple of 2

$p = 2a$ for some integer a.

$(2a)^3 = 2q^3$

$8a^3 = 2q^3$

$q^3 = 4a^3 = 2(2a^3)$

Then q^3 is a multiple of 2 and, by $(*)$, q must also be a multiple of 2.

This contradicts the assumption that p and q have no common factors greater than 1.

Conclusion:

The assumption that $\sqrt[3]{2} = \frac{p}{q}$ for some integers p and q with $\gcd(p, q) = 1$ is shown to lead to a contradiction.

Therefore, $\sqrt[3]{2}$ cannot be expressed as a ratio of integers in simplest terms.

Therefore, $\sqrt[3]{2}$ is irrational.

5

Proposition: If three real values x, y, z have mean 126 then at least one of them must be at least 126.

Assume the contrary: Suppose that the three values are ordered $x \le y \le z$ and have mean 126 but $z < 126$.

Then $x \le y \le z < 126$

So $x < 126, y < 126, z < 126$

Then $(x + y + z) < 3 \times 126$

$\frac{x + y + z}{3} < 126$

This contradicts the assumption that the mean of the three values is 126.

Conclusion:

The assumption that all three values can be below the mean of their values is shown to lead to a contradiction.

Therefore, at least one of the values must be at least 126.

In context, if the three children have mean height 126 cm then at least one of them must be at least 126 cm tall.

6 **a**

For integers a, b, c, d with $b, d \neq 0$, $\frac{a}{b}$ and $\frac{c}{d}$ are arbitrary rational values.

$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$

$(ad - bc)$ and bd are both integers, with $bd \neq 0$ so $\frac{ad-bc}{bd}$ is a rational value.

b

Proposition: If x is a rational value and y is an irrational value then $(x + y)$ is irrational.

Assume the contrary: Suppose that there is a rational value x and irrational value y for which $x + y$ is rational.

Then $(x + y) - x$ is the difference between two rational values

By part **a**, $(x + y) - x$ must be rational

That is, y must be rational.

This contradicts the assumption that y is irrational.

Conclusion:
The assumption that the sum of a rational and an irrational value can be rational is shown to lead to a contradiction.
Therefore, the sum of an irrational and a rational value must be irrational.

7
It is assumed that $\log_2 3 > \log_2 2 = 1$ so without further working, $\log_2 3$ is assumed to be positive.
Proposition: $\log_2 3$ is irrational
Assume the contrary: Suppose that $\log_2 3$ can be expressed as $\frac{p}{q}$ for some positive integers p and q
$$\log_2 3 = \frac{p}{q}$$
Putting each side as a power of 2:
$$2^{\log_2 3} = 3 = 2^{\frac{p}{q}}$$
q is a positive integer
$$3^q = \left(2^{\frac{p}{q}}\right)^q = 2^p$$
Every positive integer power of 3 must be odd, as a product of odd numbers (proof analogous to Q1)
Every positive integer power of 2 must be even, as a product of even numbers (proof given in Q1)
No odd number can equal an even number, so 3^q cannot equal 2^p.
This contradicts the assumption that $\log_2 3 = \frac{p}{q}$
Conclusion:
The assumption that $\log_2 3 = \frac{p}{q}$ for some integers p and q is shown to lead to a contradiction.
Therefore, $\log_2 3$ is irrational.

8
The proof uses the fact $(*)$ that if two integers a and b have product ab which is a multiple of 3 then either a or b must also be a multiple of 3.
Proving $(*)$ first, by contradiction:
Proposition: For integers a and b, if $ab = 3m$ for some integer m then either $a = 3n$ or $b = 3n$ for some integer n
Assume the contrary: Suppose that $ab = 3m$ and $a = 3x + r$ and $b = 3y + s$ where r and s are integers $1 \le r, s \le 2$ so that a and b are not multiples of 3.
$$\begin{aligned} \text{So } ab &= (3x + r)(3y + s) \\ &= 9xy + 3xs + 3yr + rs \\ &= 3(3xy + xs + yr) + rs \end{aligned}$$
$rs = 1, 2$ or 4, none of which are multiples of 3
So rs is not a multiple of 3, and so ab is not a multiple of 3 which contradicts the assumption.
Conclusion:
The assumption that, for integers a and b, ab can be a multiple of 3 while neither a nor b is a multiple of 3 is shown to lead to a contradiction.
Therefore, if ab is a multiple of 3, either a or b must also be a multiple of 3.

Now to the question asked:
Proposition: $\log_3 7$ is irrational
Assume the contrary: Suppose that $\log_3 7$ can be expressed as $\frac{p}{q}$ for some positive integers p and q
$\log_3 7 = \frac{p}{q}$
Putting each side as a power of 3:
$3^{\log_3 7} = 7 = 3^{\frac{p}{q}}$
q is a positive integer
$7^q = \left(3^{\frac{p}{q}}\right)^q = 3^p$
3^p for positive integer p is a multiple of 3
7 is not a multiple of 3.
$7^q = 7 \times 7^{q-1}$ and, by $(*)$, if 7^q is a multiple of 3 then, since 7 is not a multiple of 3, 7^{q-1} must be.
This argument can be iterated to prove that 7^q cannot be a multiple of 3 for any positive integer q.
Therefore, 7^q cannot equal 3^p for positive integers p and q.
Conclusion:
The assumption that $\log_3 7 = \frac{p}{q}$ for some integers p and q is shown to lead to a contradiction.
Therefore, $\log_3 7$ is irrational.

9
An even number is defined as an integer multiple of 2.
Proposition: There is no largest even integer
Assume the contrary: Suppose that there is a value N which is the largest even integer.
Then $N = 2M$ for some integer M
If M is an integer then $M + 1$ is an integer, and so is $2(M + 1)$ [since the integers are closed over addition and multiplication]
But by the definition of an even number, $2(M + 1)$ is an even number.
$2(M + 1) = 2M + 2 = N + 2 > N$
This contradicts the assumption that N is the largest even number.
Conclusion:
The assumption that there is a value N which is the largest even number is shown to lead to a contradiction.
Therefore, there is no largest even number.

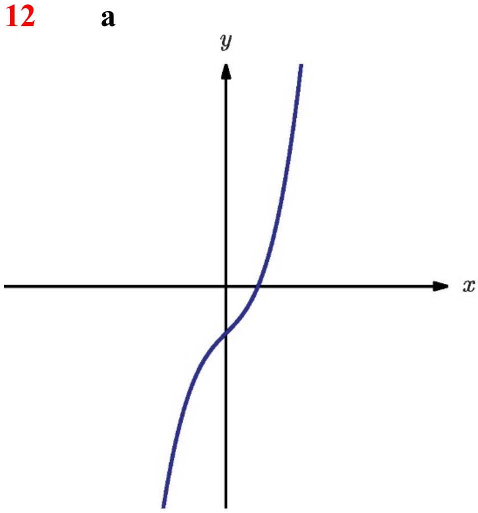
10
Proposition: There is no smallest positive real number
Assume the contrary: Suppose that there is a value d which is the smallest positive real number.
Then $d > 0$
But $0.5d$ must also be a positive value, since the product of two positive values is always positive.
For a positive value d, $0.5d < d$
So $0.5d$ is a positive real value less than d.
This contradicts the assumption that d is the smallest positive real number.

Conclusion:
The assumption that there is a value d which is the smallest positive real number is shown to lead to a contradiction.
Therefore, there is no smallest positive real number.

11 **a**
If p_1, p_2 and p_3 are prime numbers then $\frac{p_1p_2p_3+1}{p_1} = p_2p_3 + \frac{1}{p_1}$ is the sum of an integer and a non-integer, which cannot be an integer.
So $p_1p_2p_3$ is not divisible by p_1.
Equivalent arguments (or symmetry) show that $p_1p_2p_3$ is not divisible by any of p_1, p_2 or p_3

b
Proposition: There are infinitely many prime numbers
Assume the contrary: Suppose that there is a finite number N of prime numbers.
Then the primes can be listed in ascending order as $p_1 = 2, p_2 = 3, \dots, p_N$ with $p_1 < p_2 < \cdots < p_N$.
Then let q be the value found when all the primes are multiplied together, and the result added to one:
$q = p_1 \times p_2 \times p_3 \times \dots \times p_N + 1$
Using the same argument as in part **a**, q is not divisible by any of the prime numbers.
But by definition, an integer which is not divisible by any prime number with a lesser value must itself be prime.
Then q is a prime and has a value greater than p_N.
This contradicts the assumption that p_N is the greatest prime value.
Conclusion:
The assumption that there is a finite number of prime numbers is shown to lead to a contradiction.
Therefore, there are infinitely many prime numbers.

12 **a**



b The proof uses the fact $(*)$ that if an integer q has cube q^3 which is a multiple of a prime value a then q must also be a multiple of a.
The proof of this follows a similar path to that shown in question 3.

Proposition: The real root to $x^3 + x + 1 = 0$ is irrational
Assume the contrary: Suppose that the real root of $x^3 + x - 1 = 0$ can be written as $x = \frac{p}{q}$ for some integers p and q, where $\gcd(p, q) = 1$

$\frac{p^3}{q^3} + \frac{p}{q} - 1 = 0$
$p^3 + pq^2 - q^3 = 0$
$q^3 = p^3 + pq^2$
$q^3 = p(p^2 + q^2)$
Let a be the least prime factor of p.
Since the left side is a multiple of p, it must also be a multiple of a.
Therefore, the right side is a multiple of a.
But if q^3 is a multiple of prime number a then q must be a multiple of prime number a (using fact $(*)$).
Therefore, prime value a is a factor of both p and q
This contradicts the assumption that p and q have no common factors greater than 1.
Conclusion:
The assumption that root x can be written as a ratio of integers in simplest form is shown to lead to a contradiction.
Therefore, the root x is irrational.

Exercise 5C

1

Proposition: $\sqrt{x^2 - 1} \equiv x - 1$
Counterexample: Let $x = 2$
Then $\sqrt{x^2 - 1} = \sqrt{3}$
and $x - 1 = 2 - 1 = 1$
$\sqrt{3} \neq 1$
Conclusion: $x = 2$ is a counterexample.

2

Proposition: $(x - y)^3 \equiv x^3 - y^3$
Counterexample: Let $x = 2, y = 1$
Then $(x - y)^3 = 1^3 = 1$
And $x^3 - y^3 = 2^3 - 1^3 = 7$
$1 \neq 7$
Conclusion: $x = 2, y = 1$ is a counterexample.

3

Proposition: $\ln(a + b) \equiv \ln a + \ln b$
Counterexample: Let $a = 1, b = 1$
Then $\ln(a + b) = \ln(2) = 0.693$
And $\ln a + \ln b = \ln(1) + \ln(1) = 0 + 0 = 0$
$0.693 \neq 0$
Conclusion: $a = b = 1$ is a counterexample.

4

Proposition: If $\frac{dy}{dx} = 2x$ then $y = x^2$
Counterexample: Let $y = x^2 + 1$
Then $\frac{dy}{dx} = 2x$
Conclusion: $y = x^2 + 1$ is a counterexample.

5

Proposition: $\sin 2x = 1 \Rightarrow x = 45°$
Counterexample: Let $x = 225°$
Then $\sin 2x = \sin 450° = 1$
Conclusion: $x = 225°$ is a counterexample.

6

Proposition: If $\frac{a}{b} = \frac{c}{d}$, then $a = b$ and $c = d$
Counterexample: Let $a = 1, b = 2, c = 2, d = 4$
Then $\frac{a}{b} = \frac{1}{2} = 0.5$
And $\frac{c}{d} = \frac{2}{4} = 0.5$
So $\frac{a}{b} = \frac{c}{d}$ although $a \neq c$ and $b \neq d$
Conclusion: $a = 1, b = 2, c = 2, d = 4$ is a counterexample.

7

Proposition: A quadrilateral with four equal sides must be a square
Counterexample: The rhombus formed by joining two equilateral triangles along a common side is an equilateral quadrilateral whose interior angles are 60° and 120°.
A square must be equiangular (all interior angles are equal, to 90°)
Conclusion: A non-square rhombus is a counterexample.

8

Proposition: $\sqrt{x^2} = x$ for all x
Counterexample: Let $x = -1$
Then $\sqrt{x^2} = \sqrt{1} = 1$
And $x = -1$
$1 \neq -1$
Conclusion: $x = -1$ is a counterexample.

9

Proposition: If ab is an integer then a and b are both integers
Counterexample: Let $a = 0.5, b = 2$
Then $ab = 1 \in \mathbb{Z}$ although b is not an integer
Conclusion: $a = 0.5, b = 2$ is a counterexample.

10 **a**

$\text{f}(1) = 1 + 1 + 11 = 13$ which is a prime
$\text{f}(2) = 4 + 2 + 11 = 17$ which is a prime
$\text{f}(3) = 9 + 3 + 11 = 23$ which is a prime

b

Proposition: $\text{f}(n)$ is prime for all $n \in \mathbb{Z}$
Counterexample: Let $n = 11$
$\text{f}(11) = 11^2 + 11 + 11 = 11 \times 13$ which, as the product of two primes cannot be prime.
Conclusion: $n = 11$ is a counterexample.

11

Tip: Note that if you use an equivalent of x and $y = 1 - x$, you then may need the proof given in exercise 5B question 6 to show that y must be irrational. Make sure you avoid any possibility of a circular argument!

Proposition: The sum of two irrational numbers is an irrational number
Counterexample: Let x be an irrational number and $y = -x$
Then x and y must both be irrational numbers
But $x + y = 0$ which is a rational number.
Conclusion: Irrational value x and $y = -x$ is a counterexample.

12

Proposition: If $x^2 > 100$ then $x > 10$
Counterexample: Let $x = -11$
Then $x^2 = 121 > 100$
But $x < 10$
Conclusion: $x = -11$ is a counterexample.

13

Proposition: If $z^4 = 1$, then $z = 1$ or -1.
Counterexample: Let $z = \mathrm{i}$
Then $z^4 = (z^2)^2 = (-1)^2 = 1$
But $z \neq 1$ or -1.
Conclusion: Complex number $z = \mathrm{i}$ is a counterexample.

14

Proposition: An irrational number raised to an irrational number is always irrational

Counterexample: Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = x^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

Then, using the laws of indices, $y = \left(\sqrt{2}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$ which is rational.
But $y = x^{\sqrt{2}}$, so if the proposition is true, either $\sqrt{2}$ or x must be rational.
$\sqrt{2}$ is known to be irrational (see proof in Worked example 5.3).
If $x = \sqrt{2}^{\sqrt{2}}$ is rational, this would itself be a contradiction of the proposition.

Conclusion: Either $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ represents a counterexample; one of them must be rational, while being an irrational number raised to an irrational number.

Tip: The wonderful thing about a proof like this is that we can prove that one of two numbers must be rational, without ever needing to know or prove which one! There are many examples of proofs in mathematics where the existence of a thing can be proved without ever finding that thing.

Mixed Practice

1

Proposition: $\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+1)(n+2)$

Base case $n = 1$: $1 \times 2 = 2 = \frac{1}{3}(1+1)(1+2)$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\displaystyle\sum_{r=1}^{k} r(r+1) = \frac{k}{3}(k+1)(k+2)$

Working towards: $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1) &= \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)\\ &= \frac{k}{3}(k+1)(k+2) + (k+1)(k+2) \quad \text{(by assumption)}\\ &= (k+1)(k+2)\left[\frac{k}{3}+1\right]\\ &= \frac{1}{3}(k+1)(k+2)(k+3)\end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

2

Proposition: $\sum_{r=1}^{n} r(3r-5) = n(n+1)(n-2)$

Base case $n = 1$: $1(3 \times 1 - 5) = -2 = 1(1+1)(1-2)$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\displaystyle\sum_{r=1}^{k} r(3r-5) = k(k+1)(k-2)$

Working towards: $\sum_{r=1}^{k+1} r(3r-5) = (k+1)(k+2)(k-1)$

$$\begin{aligned}\sum_{r=1}^{k+1} r(3r-5) &= \sum_{r=1}^{k} r(3r-5) + (k+1)(3(k+1)-5)\\ &= \sum_{r=1}^{k} r(3r-5) + (k+1)(3k-2)\\ &= k(k+1)(k-2) + (k+1)(3k-2) \quad \text{(by assumption)}\\ &= (k+1)[k(k-2)+3k-2]\\ &= (k+1)[k^2+k-2]\\ &= (k+1)(k+2)(k-1)\end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

3

The proof uses the fact $(*)$ that if an integer k has square k^2 which is a multiple of 3 then k must also be a multiple of 3.

Proving $(*)$ first, by contradiction:

Proposition: For integer k, if $k^2 = 3m$ for some integer m then $k = 3n$ for some integer n,

Assume the contrary: Suppose that $k^2 = 3n$ and $k = 3n + r$ where r is an integer $1 \leq r \leq 2$.

So $k^2 = (3n + r)(3n + r)$

$= 9n^2 + 6nr + r^2$

$= 3(3n^2 + 2nr) + r^2$

$r^2 = 1$ or 4, neither of which are multiples of 3.

So k^2 is not a multiple of 3, which contradicts the assumption.

Conclusion:

The assumption that, for an integer k, k^2 can be a multiple of 3 while k is not a multiple of 3 is shown to lead to a contradiction.

Therefore, if k^2 is a multiple of 3, k must also be a multiple of 3.

Now to the question asked:

Proposition: $\sqrt{3}$ is irrational

Assume the contrary: Suppose that $\sqrt{3}$ can be expressed as $\frac{p}{q}$ for some integers p and q which have no common factors greater than 1 (i.e. the fraction is expressed in simplified form).

Squaring both sides:

$3 = \frac{p^2}{q^2}$

$p^2 = 3q^2$

Then p^2 is a multiple of 3 and, by $(*)$, p must also be a multiple of 3.

$p = 3a$ for some integer a.

$(3a)^2 = 3q^2$

$9a^2 = 3q^2$

$q^2 = 3a^2$

Then q^2 is a multiple of 3 and, by $(*)$, q must also be a multiple of 3.

This contradicts the assumption that p and q have no common factors greater than 1.

Conclusion:

The assumption that $\sqrt{3} = \frac{p}{q}$ for some integers p and q with $\gcd(p, q) = 1$ is shown to lead to a contradiction.

Therefore, $\sqrt{3}$ cannot be expressed as a ratio of integers in simplest terms.

Therefore, $\sqrt{3}$ is irrational.

4

Proposition: $(x + 2)^2 \equiv x^2 + 4$

Counterexample: Let $x = 1$

Then $(x + 2)^2 = (1 + 2)^2 = 9$

and $x^2 + 4 = 1 + 4 = 5$

$9 \neq 5$

Conclusion: $x = 1$ is a counterexample.

5

Proposition: $a^x + a^y \equiv a^{x+y}$
Counterexample: Let $a = 1$
Then $a^x + a^y = 1^x + 1^y = 1 + 1 = 2$
and $a^{x+y} = 1^{x+y} = 1$
$2 \neq 1$
Conclusion: $a = 1$ is a counterexample, for any values x and y.

6

Proposition: If $a + b$ is an integer then a and b are both integers
Counterexample: Let $a = 0.5, b = 0.5$
Then $a + b = 1 \in \mathbb{Z}$
But $a, b \notin \mathbb{Z}$
Conclusion: $a = b = 0.5$ is a counterexample.

7

Proposition: All prime numbers are odd
Counterexample: 2 is a prime number and is even
Conclusion: 2 is a (the only) counterexample.

8

Proposition: $\sum_{r=1}^{n} r(r+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$

Base case $n = 1$: $1(1+1)^2 = 4 = \frac{1(1+1)(1+2)(3+5)}{12}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\displaystyle\sum_{r=1}^{k} r(r+1)^2 = \frac{k(k+1)(k+2)(3k+5)}{12}$

Working towards: $\sum_{r=1}^{k+1} r(r+1)^2 = \frac{(k+1)(k+2)(k+3)(3k+8)}{12}$

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1)^2 &= \sum_{r=1}^{k} r(r+1)^2 + (k+1)(k+2)^2 \\ &= \frac{k(k+1)(k+2)(3k+5)}{12} + (k+1)(k+2)^2 \quad \text{(by assumption)} \\ &= \frac{(k+1)(k+2)}{12}[k(3k+5) + 12(k+2)] \\ &= \frac{(k+1)(k+2)}{12}[3k^2 + 17k + 24] \\ &= \frac{(k+1)(k+2)}{12}(k+3)(3k+8)\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

9

Proposition: $\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)} = \frac{2n}{2n+1}$

Base case $n = 1$: $\frac{2}{(2-1)(2+1)} = \frac{2}{3} = \frac{2\times 1}{2\times 1+1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k} \frac{2}{(2r-1)(2r+1)} = \frac{2k}{2k+1}$

Working towards: $\sum_{r=1}^{k+1} \frac{2}{(2r-1)(2r+1)} = \frac{2(k+1)}{2k+3}$

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{2}{(2r-1)(2r+1)} &= \sum_{r=1}^{k} \frac{2}{(2r-1)(2r+1)} + \frac{2}{(2k+1)(2k+3)} \\ &= \frac{2k}{2k+1} + \frac{2}{(2k+1)(2k+3)} \quad \text{(by assumption)} \\ &= \frac{1}{(2k+1)(2k+3)}[2k(2k+3)+2] \\ &= \frac{1}{(2k+1)(2k+3)}[4k^2+6k+2] \\ &= \frac{2}{(2k+1)(2k+3)}[2k^2+3k+1] \\ &= \frac{2}{(2k+1)(2k+3)}(2k+1)(k+1) \\ &= \frac{2(k+1)}{(2k+3)}\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

10

Proposition: $12^n - 1 = 11m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $12^0 - 1 = 0 = 11 \times 0$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $12^k - 1 = 11A$ for some $A \in \mathbb{Z}$

Working towards: $12^{k+1} - 1 = 11B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}12^{k+1} - 1 &= 12(12^k) - 1 \\ &= 12(12^k - 1) + 11 \\ &= 12(11A) + 11 \text{(by assumption)} \\ &= 11(12A + 1) \\ &= 11B \text{(where } B = 12A + 1 \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11

<u>Proposition</u>: $3^{2n} + 7 = 8m$ for some $m \in \mathbb{Z}$

<u>Base case $n = 0$</u>: $3^0 + 7 = 8 = 1 \times 8$ so the proposition is true for $n = 0$

<u>Inductive step:</u> Assume the proposition is true for integer $n = k \geq 0$

So $3^{2k} + 7 = 8A$ for some $A \in \mathbb{Z}$

<u>Working towards:</u> $3^{2(k+1)} + 7 = 8B$ for some $B \in \mathbb{Z}$

$$\begin{aligned} 3^{2(k+1)} - 1 &= 9(3^{2k}) + 7 \\ &= 9(3^{2k} + 7) - 56 \\ &= 9(8A) - 56 (\text{by assumption}) \\ &= 8(9A - 7) \\ &= 8B (\text{where } B = 9A - 7 \in \mathbb{Z}) \end{aligned}$$

So the proposition is true for $n = k + 1$

<u>Conclusion</u>:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12

<u>Proposition</u>: $5^{2n} - 24n - 1 = 576m$ for some $m \in \mathbb{Z}$

<u>Base case $n = 0$</u>: $5^0 - 24(0) - 1 = 0 = 576 \times 0$ so the proposition is true for $n = 0$

<u>Inductive step:</u> Assume the proposition is true for integer $n = k \geq 0$

So $5^{2k} - 24k - 1 = 576A$ for some $A \in \mathbb{Z}$

<u>Working towards:</u> $5^{2(k+1)} - 24(k + 1) - 1 = 576B$ for some $B \in \mathbb{Z}$

$$\begin{aligned} 5^{2(k+1)} - 24(k+1) - 1 &= 25(5^{2k}) - 24k - 25 \\ &= 25(5^{2k} - 24k - 1) + 24(24k) \\ &= 5(576A) + 576k (\text{by assumption}) \\ &= 576(5A + k) \\ &= 576B (\text{where } B = 5A + k \in \mathbb{Z}) \end{aligned}$$

So the proposition is true for $n = k + 1$

<u>Conclusion</u>:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 **a**

$$\mathrm{f}(x) = \frac{x}{2 - x}$$

$$\begin{aligned} (\mathrm{f} \circ \mathrm{f})(x) &= \frac{\left(\frac{x}{2 - x}\right)}{2 - \left(\frac{x}{2 - x}\right)} \\ &= \frac{x}{2(2 - x) - x} \\ &= \frac{x}{4 - 3x} \end{aligned}$$

b

Proposition: $(\mathrm{f} \circ \ldots \circ \mathrm{f})(x) = \frac{x}{2^n-(2^n-1)x}$

Base case $n = 1$: $\mathrm{f}(x) = \frac{x}{2-x} = \frac{x}{2^1-(2^1-1)x} = \mathrm{F}_1(x)$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $\underbrace{(\mathrm{f} \circ \cdots \circ \mathrm{f})}_{k \text{ times}}(x) = \dfrac{x}{2^k - (2^k - 1)x}$

Working towards:

$\underbrace{(\mathrm{f} \circ \cdots \circ \mathrm{f})}_{(k+1) \text{ times}}(x) = \dfrac{x}{2^{k+1} - (2^{k+1} - 1)x}$

$$\underbrace{(\mathrm{f} \circ \cdots . \circ \mathrm{f})}_{(k+1) \text{ times}}(x) = \mathrm{f}(\underbrace{\mathrm{f} \circ \cdots \circ \mathrm{f}}_{k \text{ times}})(x)$$

$$= \mathrm{f}\left(\frac{x}{2^k - (2^k - 1)x}\right) \quad \text{(by assumption)}$$

$$= \frac{\dfrac{x}{2^k - (2^k - 1)x}}{2 - \left(\dfrac{x}{2^k - (2^k - 1)x}\right)}$$

$$= \frac{x}{2(2^k - (2^k - 1)x) - x}$$

$$= \frac{x}{2^{k+1} - (2^{k+1} - 2)x - x}$$

$$= \frac{x}{2^{k+1} - (2^{k+1} - 1)x}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

c

$$\mathrm{F}_n(x) = \frac{x}{2^n - (2^n - 1)x}$$

$$\mathrm{F}_{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x} = \frac{2^n x}{1 + (2^n - 1)x}$$

Then $$\mathrm{F}_n\big(\mathrm{F}_{-n}(x)\big) = \frac{\dfrac{2^n x}{1 + (2^n - 1)x}}{2^n - (2^n - 1)\left(\dfrac{2^n x}{1 + (2^n - 1)x}\right)}$$

$$= \frac{2^n x}{2^n(1 + (2^n - 1)x) - (2^n - 1)2^n x}$$

$$= \frac{2^n x}{2^n + 2^{2n}x - 2^n x - 2^{2n}x + 2^n x}$$

$$= x$$

That is, $\mathrm{F_n}(x)$ is the inverse of $\mathrm{F}_{-n}(x)$ (and vice versa, by replacing n with $-n$ throughout).

14 **ai**

$$\sum_{i=1}^{n}(2i-1)$$

aii Proof by induction:

Proposition:

$$\sum_{i=1}^{n}(2i-1)=n^2$$

Base case $n=1$:

$1=1^2$ so the proposition is true for $n=1$

Inductive step:

Assume the proposition is true for integer $n=k\geq 1$

So $\sum_{i=1}^{k}(2i-1)=k^2$

Working towards: $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1)&=\sum_{i=1}^{k}(2i-1)+(2(k+1)-1)\\&=k^2+(2k+1)\text{(by assumption)}\\&=(k+1)^2\end{aligned}$$

So the proposition is true for $n=k+1$

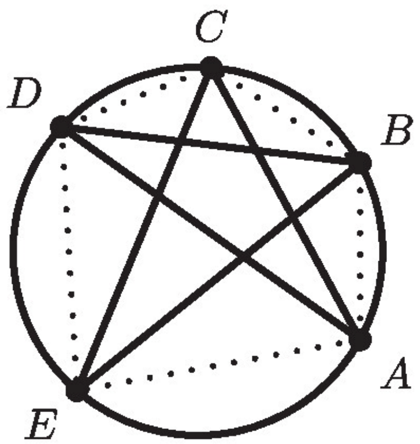
Conclusion:

The proposition is true for $n=1$, and, if true for $n=k$, it is also true for $n=k+1$. Therefore, the proposition is true for all $n\in\mathbb{Z}^+$ by the principle of mathematical induction.

aiii

$$\sum_{i=1}^{47}(2i-1)-\sum_{i=1}^{14}(2i-1)=47^2-14^2=2013$$

bi



bii

Each of the n points is connected to each of the other $(n-1)$ points.

Of those line segments, two will be edges of the polygon (connecting to the consecutive vertex on each side) and the remaining $n-3$ will be diagonals.

In the sum $n(n-3)$, each diagonal will be counted twice (diagonal AC will be counted both from A and from C), so the total number of diagonals when each counted once must be $\frac{n(n-3)}{2}$.

biii

Require $\frac{n(n-3)}{2} > 10^6$

$n^2 - 3n - 2000000 > 0$

A positive quadratic has values greater than 0 for values outside the roots.

The only positive root of the quadratic is $n = \frac{3+\sqrt{8000009}}{2} \approx 1415.7$

The least such integer solution for $n > 1415.7$ is $n = 1416$

15

The i^{th} term of a geometric series with first term a and common ratio r is ar^{i-1}.

The working and the formula to prove presumes that $r \neq 1$.

Proposition:

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n - 1)}{r - 1}$$

Base case $n = 1$:

$a = \frac{a(r-1)}{r-1}$ so the proposition is true for $n = 1$

Inductive step:

Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{i=1}^{k} ar^{i-1} = \frac{a(r^k - 1)}{r - 1}$

Working towards:

$$\sum_{i=1}^{k+1} ar^{i-1} = \frac{a(r^{k+1} - 1)}{r - 1}$$

$$\begin{aligned}\sum_{i=1}^{k+1} ar^{i-1} &= \sum_{i=1}^{k} ar^{i-1} + ar^k \\ &= \frac{a(r^k - 1)}{r - 1} + ar^k \text{(by assumption)} \\ &= \frac{a}{r-1}(r^k - 1 + (r-1)r^k) \\ &= \frac{a}{r-1}(r^{k+1} - 1)\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

16

Proposition: $9^n - 2^n = 7m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $9^0 - 2^0 = 0 = 7 \times 0$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $9^k - 2^k = 7A$ for some $A \in \mathbb{Z}$

Working towards: $9^{k+1} - 2^{k+1} = 7B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}9^{k+1} - 2^{k+1} &= 9(9^k) - 2(2^k)\\ &= 9(9^k - 2^k) + 7(2^k)\\ &= 9(7A) + 7(2^k)\text{(by assumption)}\\ &= 7(9A + 2^k)\\ &= 7B\text{(where } B = 9A + 2^k \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

17

Proposition: $15^n - 2^n = 13m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $15^0 - 2^0 = 0 = 13 \times 0$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $15^k - 2^k = 13A$ for some $A \in \mathbb{Z}$

Working towards: $15^{k+1} - 2^{k+1} = 13B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}15^{k+1} - 2^{k+1} &= 15(15^k) - 2(2^k)\\ &= 15(15^k - 2^k) + 13(2^k)\\ &= 15(13A) + 13(2^k)\text{(by assumption)}\\ &= 13(15A + 2^k)\\ &= 13B\text{(where } B = 15A + 2^k \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

18

Proposition: $11^{n+2} + 12^{2n+1} = 133m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $11^2 + 12^1 = 133 = 133 \times 1$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $11^{k+2} + 12^{2k+1} = 133A$ for some $A \in \mathbb{Z}$

Working towards: $11^{k+3} + 12^{2k+3} = 133B$ for some $B \in \mathbb{Z}$

$$\begin{aligned}11^{k+3} + 12^{2k+3} &= 11(11^{k+2}) + 144(12^{2k+1})\\ &= 11(11^{k+2} + 12^{2k+1}) + 133(12^{2k+1})\\ &= 11(133A) + 133(12^{2k+1})\text{(by assumption)}\\ &= 133(11A + 12^{2k+1})\\ &= 133B\text{(where } B = 11A + 12^{2k+1} \in \mathbb{Z})\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

19

Proposition: $\log_{10} 3$ is an irrational number

Assume the contrary: Suppose that $\log_{10} 3$ is rational.

Then $\log_{10} 3 = \frac{p}{q}$ for some integer values p and q, where $q \neq 0$

Since $3 > 1$, $\log_{10} 3 > 0$ so it can be assumed that both p and q are positive values.

So $3 = 10^{\frac{p}{q}}$

$3^q = 10^p$

3 is odd, so any positive integer power of 3 must also be odd.

10 is even, so any positive integer power of 10 must also be even.

Then 3^q cannot equal 10^p, since no integer can be both odd and even.

This contradicts the assumption that there are positive integer values p and q with this property.

Conclusion:

The assumption that $\log_{10} 3$ is rational is shown to lead to a contradiction.

Therefore, $\log_{10} 3$ is irrational.

20 **a**

Proposition: For integers a and b, if ab is odd then both a and b must be odd

Assume the contrary: Suppose that ab is odd and a is even.

Then $a = 2k$ for some integer k

So $ab = 2kb$

$= 2(kb)$

$= 2m$ for some integer m

So ab is even, which contradicts the assumption.

Assuming b is even follows exactly the same reasoning, to the same contradiction.

Conclusion:

The assumption that if ab is odd then a or b could be even is shown to lead to a contradiction.

Therefore, if ab is odd, both a and b must also be odd.

b

Proposition: If ab is even then both a and b must be even

Counterexample: Let $a = 2, b = 3$

Then $ab = 6$ which is even as required

But b is odd

Conclusion: $a = 2, b = 3$ is a counterexample.

21

Proposition: There are infinitely many odd numbers

Assume the contrary: Suppose that there are only finitely many odd numbers.

Then there must be a greatest odd number, N, since the odd numbers can be listed in ascending order.

Then $N + 1$ must be an even number, so $N + 1 = 2k$ for some integer k

But then $N + 2 = 2k + 1$ is an integer which is not a multiple of 2.

$N + 2$ is therefore an odd number, which is greater than N, which contradicts the assumption that N is the greatest odd number.

Conclusion:

The assumption that there are finitely many odd numbers is shown to lead to a contradiction.

Therefore, there are infinitely many odd numbers.

22

<u>Proposition</u>: Two straight lines which do not intersect must be parallel

<u>Counterexample</u>:

Let l_1 be the line given by vector equation $\mathbf{r} = \lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ (this is the x-axis)

Let l_2 be the line given by vector equation $\mathbf{r} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\0\\1\end{pmatrix}$ (this is a line running parallel to the z axis)

The two lines do not intersect, since the y-coordinate of any point on l_1 is 0 and the y-coordinate of any point on l_2 is 1.

But the lines are not parallel – in this example, the direction vectors are perpendicular.

<u>Conclusion</u>: If the lines can be in three (or more) dimensions then they can be skew – neither intersecting nor parallel.

23

<u>Proposition</u>:

$$\sum_{r=1}^{n}\frac{r}{2^r} = 2-(n+2)\left(\frac{1}{2}\right)^n$$

<u>Base case $n = 1$:</u>

$\frac{1}{2^1} = \frac{1}{2} = 2-(1+2)\left(\frac{1}{2}\right)$ so the proposition is true for $n = 1$

<u>Inductive step:</u>

Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{r=1}^{k}\frac{r}{2^r} = 2-(k+2)\left(\frac{1}{2}\right)^k$

<u>*Working towards:*</u>

$$\sum_{r=1}^{k+1}\frac{r}{2^r} = 2-(k+3)\left(\frac{1}{2}\right)^{k+1}$$

$$\sum_{r=1}^{k+1}\frac{r}{2^r} = \sum_{r=1}^{k}\frac{r}{2^r} + \frac{k+1}{2^{k+1}}$$

$$= 2-(k+2)\left(\frac{1}{2}\right)^k + \frac{k+1}{2^{k+1}} \text{(by assumption)}$$

$$= 2-\left(\frac{1}{2}\right)^{k+1}(2(k+2)-(k+1))$$

$$= 2-\left(\frac{1}{2}\right)^{k+1}(k+3)$$

So the proposition is true for $n = k+1$

<u>Conclusion</u>:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

24

Proposition: $2^n > 11n$ for $n \geq 7$

Base case $n = 7$: $2^7 = 128 > 77$ so the proposition is true for $n = 7$

Inductive step: Assume the proposition is true for integer $n = k \geq 7$

So $2^k > 11k$

Working towards: $2^{k+1} > 11(k+1)$

$$\begin{aligned} 2^{k+1} &= 2(2^k) \\ &> 2(11k)\text{(by assumption)} \\ &> 11k + 11 \text{ for } k > 1 \\ &> 11(k+1) \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 7$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 7, n \in \mathbb{Z}$ by the principle of mathematical induction.

25 **a**

$$\begin{aligned} \frac{1}{\sqrt{n}+\sqrt{n+1}} &= \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} \\ &= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n} \\ &= \sqrt{n+1}-\sqrt{n} \end{aligned}$$

b

Substituting $n = 1$ into the result from part **a**:

$$\sqrt{2}-\sqrt{1} = \frac{1}{\sqrt{1}+\sqrt{2}}$$

$$\sqrt{2}-1 = \frac{1}{1+\sqrt{2}} < \frac{1}{\sqrt{2}}$$

c

Proposition:

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > \sqrt{n}$$

Base case $n = 2$:

$1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true from part **b**

So the proposition is true for $n = 2$

Inductive step:

Assume the proposition is true for integer $n = k \geq 2$

So $\sum_{r=1}^{k} \frac{1}{\sqrt{r}} > \sqrt{k}$

Working towards:

$$\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} > \sqrt{k+1}$$

From part **a**:

$$\sqrt{k+1}-\sqrt{k} = \frac{1}{\sqrt{k}+\sqrt{k+1}} < \frac{1}{\sqrt{k+1}}$$

So $\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k} (*)$

$$\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} = \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}}$$
$$> \sqrt{k} + \frac{1}{\sqrt{k+1}} \text{(by assumption)}$$
$$> \sqrt{k} + \left(\sqrt{k+1} - \sqrt{k}\right)(\text{by } (*))$$
$$> \sqrt{k+1}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 2$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 2, n \in \mathbb{Z}$ by the principle of mathematical induction.

26

Proposition: $(2n)! \geq 2^n (n!)^2$ for $n \in \mathbb{Z}^+$

Base case $n = 1$: $(2 \times 1)! = 2! = 2 = 2^1(1!)^2$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $(2k)! \geq 2^k (k!)^2$

Working towards: $(2k+2)! \geq 2^{(k+1)}\left((k+1)!\right)^2$

$$(2k+2)! = (2k+2)(2k+1) \times (2k)!$$
$$> (2k+2)(2k+1) \times 2^k (k!)^2 \text{(by assumption)}$$
$$> 2(k+1)(2k+1) \times 2^k (k!)^2 \text{(taking factor of 2)}$$
$$> 2(k+1)(k+1) \times 2^k (k!)^2 (2k+1 > k+1 \text{ for } k \geq 1)$$
$$> 2(k+1)^2 \times 2^k (k!)^2$$
$$> 2^{k+1}\left((k+1)!\right)^2$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 1, n \in \mathbb{Z}$ by the principle of mathematical induction.

27 **a**

Proposition: $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for $n \in \mathbb{Z}^+$

Base case $n = 1$: $\cos\theta + \mathrm{i}\sin\theta = \cos\theta + \mathrm{i}\sin\theta$, trivially, so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$

Working towards: $(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = \cos\left((k+1)\theta\right) + \mathrm{i}\sin\left((k+1)\theta\right)$

$$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$$
$$= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)\text{(by assumption)}$$
$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}\,(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$
$$= \cos\left((k+1)\theta\right) + \mathrm{i}\sin\left((k+1)\theta\right) \text{ (compound angle formulae)}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

b

Proposition: $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for $n \in \mathbb{Z}^-$

Base case $n = -1$:

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{-1} &= \frac{1}{\cos\theta + \mathrm{i}\sin\theta} \\ &= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta + \sin^2\theta} \\ &= \cos\theta - \mathrm{i}\sin\theta \\ &= \cos(-\theta) + \mathrm{i}\sin(-\theta)\end{aligned}$$

so the proposition is true for $n = -1$

Inductive step: Assume the proposition is true for integer $n = -k \le -1$

So $(\cos\theta + \mathrm{i}\sin\theta)^{-k} = \cos(-k\theta) + \mathrm{i}\sin(-k\theta)$

Working towards: $(\cos\theta + \mathrm{i}\sin\theta)^{-k-1} = \cos\big((-k-1)\theta\big) + \mathrm{i}\sin\big((-k-1)\theta\big)$

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{-k-1} &= (\cos\theta + \mathrm{i}\sin\theta)^{-k}(\cos\theta + \mathrm{i}\sin\theta)^{-1} \\ &= (\cos(-k\theta) + \mathrm{i}\sin(-k\theta))(\cos\theta + \mathrm{i}\sin\theta)^{-1} \text{(by assumption)} \\ &= (\cos(-k\theta) + \mathrm{i}\sin(-k\theta))(\cos(-\theta) \\ &\quad + \mathrm{i}\sin(-\theta)) \text{(by base case)} \\ &= \cos(-k\theta)\cos(-\theta) - \sin(-k\theta)\sin(-\theta) \\ &\quad + \mathrm{i}\,(\sin(-k\theta)\cos(-\theta) + \cos(-k\theta)\sin(-\,\theta)) \\ &= \cos\big((-k-1)\theta\big) \\ &\quad + \mathrm{i}\sin\big((-k-1)\theta\big) \text{ (compound angle formulae)}\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = -1$, and, if true for $n = -k$, it is also true for $n = -k - 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^-$ by the principle of mathematical induction.

An alternative is given below, for those who would prefer to avoid duplicating the induction in part **a**.

Proposition: $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for $n \in \mathbb{Z}^-$

Since $(\cos\theta - \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta) = \cos^2\theta + \sin^2\theta = 1$,

it follows that $(\cos\theta + \mathrm{i}\sin\theta)^{-1} = (\cos\theta - \mathrm{i}\sin\theta) = (\cos(-\theta) + \mathrm{i}\sin(-\theta))$

Let $n \in \mathbb{Z}^{-1}$ and $m = -n \in \mathbb{Z}^+$

$$\begin{aligned}\text{Then } (\cos\theta + \mathrm{i}\sin\theta)^n &= (\cos\theta + \mathrm{i}\sin\theta)^{-m} \\ &= ((\cos\theta + \mathrm{i}\sin\theta)^{-1})^m \\ &= (\cos(-\theta) + \mathrm{i}\sin(-\theta))^m \\ &= \cos(-m\theta) \\ &\quad + \mathrm{i}\sin(-m\theta) \text{ (by induction in part } \mathbf{a}\text{, since } m \in \mathbb{Z}^+) \\ &= \cos n\theta + \mathrm{i}\sin n\theta\end{aligned}$$

Conclusion:

The proposition is true for all $n \in \mathbb{Z}^-$ by adapting the proof by induction of part a.

28

If the formula is accurate, then the sum for $n = 1$ would be $1^2 = \frac{1}{3}(a-1)$ so $a = 4$

Proposition:

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{n}{3}(4n^2-1)$$

Base case $n = 1$:

$1^2 = \frac{1}{3}(4-1)$ so the proposition is true for $n = 1$

Inductive step:

Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{i=1}^{k}(2i-1)^2 = \frac{k}{3}(4k^2-1)$

Working towards:

$$\sum_{i=1}^{k+1}(2i-1)^2 = \frac{k+1}{3}(4(k+1)^2-1)$$

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1)^2 &= \sum_{i=1}^{k}(2i-1)^2 + (2(k+1)-1)^2 \\ &= \sum_{i=1}^{k}(2i-1)^2 + (2k+1)^2 \\ &= \frac{k}{3}(4k^2-1) + (2k+1)^2 \text{(by assumption)} \\ &= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2 \\ &= \frac{2k+1}{3}(2k^2-k+6k+3) \\ &= \frac{2k+1}{3}(2k^2+5k+3) \\ &= \frac{2k+1}{3}(2k+3)(k+1) \\ &= \frac{k+1}{3}(2k+1)(2k+3) \\ &= \frac{k+1}{3}(4k^2+8k+3) \\ &= \frac{(k+1)}{3}(4(k+1)^2-1)\end{aligned}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

29

The product notation is used in the working below. Just as the sum notation uses the upper case letter 'S' from the Greek alphabet (sigma), there is a similar notation for product which uses the upper case 'P' (pi) to indicate that the indexed terms should be multiplied together.

Using this notation, we could write $n!$ as the result of a product:

$$\prod_{i=1}^{n} i = 1 \times 2 \times \ldots \times n = n!$$

This notation is not taught within the IB course and it is expected that most students would use the ellipsis (three dots: …) to indicate a continued pattern instead. However, a compact and exactly defined product can keep the working clearer, and this notation is as standard as the sum notation in mathematical literature.

Proposition:

$$\prod_{i=1}^{n}(4i-2) = \frac{(2n)!}{n!}$$

Base case $n = 1$:

$4(1) - 2 = 2 = \frac{(2\times 1)!}{1!}$ so the proposition is true for $n = 1$

Inductive step:

Assume the proposition is true for integer $n = k \geq 1$

So $\prod_{i=1}^{k}(4i-2) = \frac{(2k)!}{k!}$

Working towards:

$$\prod_{i=1}^{k+1}(4i-2) = \frac{(2k+2)!}{(k+1)!}$$

$$\begin{aligned}\prod_{i=1}^{k+1}(4i-2) &= \left(\prod_{i=1}^{k}(4i-2)\right) \times (4(k+1)-2) \\ &= \left(\prod_{i=1}^{k}(4i-2)\right) \times 2(2k+1) \\ &= \frac{(2k)!}{k!} \times 2(2k+1) \text{(by assumption)} \\ &= \frac{(2k)!}{k!} \times \frac{2(2k+1)(k+1)}{k+1} \\ &= \frac{(2k)!}{k!} \times \frac{(2k+1)(2k+2)}{k+1} \\ &= \frac{(2k+2)!}{(k+1)!}\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

30

Proposition:

$$\prod_{i=0}^{n}\cos(2^i x) = \frac{\sin(2^{n+1}x)}{2^{n+1}\sin x} \text{ (for } x \neq n\pi, n \in \mathbb{Z})$$

Base case $n = 0$:

The double angle formula for sine gives $\sin 2A = 2\sin A\cos A$

So $\cos A = \frac{\sin(2A)}{2\sin A}$ $\quad (*)$

Substituting $A = x$, $\cos x = \frac{\sin x\cos x}{\sin x} = \frac{\sin(2x)}{2\sin x} = \frac{\sin(2^{0+1}x)}{2^{0+1}\sin x}$ so the proposition is true for $n = 0$

Inductive step:

Assume the proposition is true for integer $n = k \geq 1$

So $\prod_{i=0}^{k}\cos(2^i x) = \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$

Working towards:

$$\prod_{i=0}^{k+1}\cos(2^i x) = \frac{\sin(2^{k+2}x)}{2^{k+2}\sin x}$$

$$\prod_{i=0}^{k+1}\cos(2^i x) = \left(\prod_{i=1}^{k}\cos(2^i x)\right)\times\cos(2^{k+1}x)$$

$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}\times\cos(2^{k+1}x) \text{ (by assumption)}$$

$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}\times\frac{\sin(2^{k+2}x)}{2\sin(2^{k+1}x)} \text{ (by } (*), \text{ with } A = 2^{k+1}x)$$

$$= \frac{\sin(2^{k+2}x)}{2^{k+2}\sin x}\times\frac{\sin(2^{k+1}x)}{\sin(2^{k+1}x)}$$

$$= \frac{\sin(2^{k+2}x)}{2^{k+2}\sin x}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}$ by the principle of mathematical induction.

31

As in many of the more complicated induction proofs, this requires a preliminary identity to be established. While it is fine to do this within the body of the induction proof, it is tidier to establish the identity first and then refer to it, if you can look ahead to see what you will need. Alternatively, add the identity proof as a footnote to your main proof.

a

The induction proof will use the relationship that $\sin 2A\sin 2B = \sin^2(A+B) - \sin^2(A-B)$ $\quad (*)$

Proof of this identity using double angle formulae:

$$\sin^2(A+B) = (\sin A\cos B + \cos A\sin B)^2$$
$$= \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + 2\sin A\cos A\sin B\cos B$$

Similarly,
$\sin^2(A-B) = \sin^2 A\cos^2 B + \cos^2 A\sin^2 B - 2\sin A\cos A\sin B\cos B$
Then $\sin^2(A+B) - \sin^2(A-B) = 4\sin A\cos A\sin B\cos B$

$$= (2\sin A\cos A)(2\sin B\cos B)$$

$$= \sin 2A\sin 2B$$

Main inductive proof argument:
Proposition:

$$\sum_{i=1}^{n}\sin\big((2i-1)\theta\big) = \frac{\sin^2(n\theta)}{\sin\theta} \text{ for } \theta \neq n\pi, n \in \mathbb{Z}^+$$

Base case $n = 1$:
$\sin\theta = \frac{\sin^2\theta}{\sin\theta}$ as long as $\sin\theta \neq 0$, so the proposition is true for $n = 1$
Inductive step:
Assume the proposition is true for integer $n = k \geq 1$

So $\displaystyle\sum_{i=1}^{k}\sin\big((2i-1)\theta\big) = \frac{\sin^2(k\theta)}{\sin\theta}$

Working towards:

$$\sum_{i=1}^{k+1}\sin\big((2i-1)\theta\big) = \frac{\sin^2\big((k+1)\theta\big)}{\sin\theta}$$

$$\sum_{i=1}^{k+1}\sin\big((2i-1)\theta\big) = \sum_{i=1}^{k}\sin\big((2i-1)\theta\big) + \sin\big((2k+1)\theta\big)$$

$$= \frac{\sin^2(k\theta)}{\sin\theta} + \sin\big((2k+1)\theta\big) \text{ (by assumption)}$$

$$= \frac{\sin^2(k\theta) + \sin\theta\sin\big((2k+1)\theta\big)}{\sin\theta}$$

Using $(*)$, setting $2A = (2k+1)\theta$ and $2B = \theta$,

Then $(A+B) = \dfrac{(2k+1)\theta + \theta}{2} = (k+1)\theta$

And $(A-B) = \dfrac{(2k+1)\theta - \theta}{2} = k\theta$

$$\sum_{i=1}^{k+1}\sin\big((2i-1)\theta\big) = \frac{\sin^2(k\theta) + \sin^2\big((k+1)\theta\big) - \sin^2(k\theta)}{\sin\theta}$$

$$= \frac{\sin^2\big((k+1)\theta\big)}{\sin\theta}$$

So the proposition is true for $n = k+1$
Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$.
Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

b

Substituting $\theta = \frac{\pi}{7}$ and $n = 7$:

$$\sum_{i=1}^{7} \sin\left(\frac{(2i-1)\pi}{7}\right) = \frac{\sin^2(\pi)}{\sin\left(\frac{\pi}{7}\right)} = 0$$

32

Proposition:

$$\sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{(n+1)!-1}{(n+1)!}$$

Base case $n = 1$:

$\frac{1}{2!} = \frac{2-1}{2!} = \frac{2!-1}{2!}$ so the proposition is true for $n = 1$

Inductive step:

Assume the proposition is true for integer $n = k \geq 1$

So $\sum_{i=1}^{k} \frac{i}{(i+1)!} = \frac{(k+1)!-1}{(k+1)!}$

Working towards:

$$\sum_{i=1}^{k+1} \frac{i}{(i+1)!} = \frac{(k+2)!-1}{(k+2)!}$$

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \sum_{i=1}^{k} \frac{i}{(i+1)!} + \frac{k+1}{(k+2)!} \\ &= \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!} \text{ (by assumption)} \\ &= \frac{1}{(k+2)!}\left((k+2)(k+1)! - (k+2) + k + 1\right) \\ &= \frac{1}{(k+2)!}\left((k+2)! - 1\right)\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

33 **a**

$${}^{n}C_r = \frac{n!}{(n-r)!\,r!}$$

So ${}^{n-1}C_{r-1} = \frac{(n-1)!}{(n-r)!\,(r-1)!}$ and ${}^{n-1}C_r = \frac{(n-1)!}{(n-r-1)!\,r!}$

Summing these two expressions:

$$\begin{aligned}{}^{n-1}C_{r-1} + {}^{n-1}C_r &= \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-r-1)!\,r!} \\ &= \frac{r(n-1)!}{(n-r)!\,r!} + \frac{(n-r)(n-1)!}{(n-r)!\,r!}\end{aligned}$$

$$= \frac{(n-1)!}{(n-r)!\,r!}(r + (n-r))$$
$$= \frac{n(n-1)!}{(n-r)!\,r!}$$
$$= \frac{n!}{(n-r)!\,r!}$$
$$= {}^nC_r$$

b

Proposition:

$$\sum_{r=1}^{n-1} {}^nC_r = 2^n - 2$$

Base case $n = 2$:

${}^2C_1 = 2 = 2^2 - 2$ so the proposition is true for $n = 2$

Inductive step:

Assume the proposition is true for integer $n = k \geq 2$

So $\sum_{r=1}^{k-1} {}^kC_r = 2^k - 2$

Working towards:

$$\sum_{r=1}^{k} {}^{k+1}C_r = 2^{k+1} - 2$$

$$\sum_{r=1}^{k} {}^{k+1}C_r = \sum_{r=1}^{k} ({}^kC_{r-1} + {}^kC_r) \text{ (from part } \mathbf{a}\text{)}$$
$$= \left(\sum_{r=1}^{k-1} {}^kC_r + {}^kC_0\right) + \left(\sum_{r=1}^{k-1} {}^kC_r + {}^kC_k\right)$$
$$= (2^k - 2 + 1) + (2^k - 2 + 1) \text{(by assumption)}$$
$$= 2(2^k - 1)$$
$$= 2^{k+1} - 2$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 2$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \geq 2, n \in \mathbb{Z}$ by the principle of mathematical induction.

34 **a**

Proposition: $n^5 - n = 5m$ for some $m \in \mathbb{Z}$

Base case $n = 0$: $0^5 - 0 = 0 = 5 \times 0$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 1$

So $k^5 - k = 5A$ for some $A \in \mathbb{Z}$

Working towards: $(k+1)^5 - (k+1) = 5B$ for some $B \in \mathbb{Z}$

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$= k^5 - k + 5k^4 + 10k^3 + 10k^2 + 5k$$
$$= k^5 - k + 5(k^4 + 2k^3 + 2k^2 + k)$$
$$= 5A + 5(k^4 + 2k^3 + 2k^2 + k) \text{(by assumption)}$$
$$= 5B \text{(where } B = A + k^4 + 2k^3 + 2k^2 + k \in \mathbb{Z}\text{)}$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b

$n^5 - n = n(n^4 - 1) = n(n^2 + 1)(n^2 - 1) = (n - 1)n(n + 1)(n^2 + 1)$

Since $n - 1, n$ and $n + 1$ are a sequence of three consecutive numbers, at least one must be even and exactly one must be a multiple of 3. Therefore, their product must be a multiple of both 2 and 3 and is therefore a multiple of 6.

c

If n is odd then both $n - 1$ and $n + 1$ (and also $n^2 + 1$) must be even, so $n^5 - n$ has 8 as a factor in addition to 3 and 5, so must be a multiple of 120 and therefore a multiple of 60.
However, if n is even then of $(n - 1)$, n, $(n + 1)$ and $(n^2 + 1)$ only n is even. The product is therefore not a multiple of 4 so cannot be a multiple of 60.
Any even value can be a counterexample:
If $n = 2$ then $n^5 - n = 32 - 2 = 30$ which is not a multiple of 60.

35

Let the number of intersection points for n lines, under the arrangement described, be N_n, and the number of regions be R_n.
Proposition 1: $N_n = \frac{n(n-1)}{2}$
Base case $n = 1$:
There are no intersection points; $N_1 = 0 = \frac{1(1-1)}{2}$ so the proposition is true for $n = 1$
Inductive step: Assume the proposition is true for any valid arrangement of k lines for some $k \geq 1$
So $N_k = \frac{k(k-1)}{2}$
Working towards: $\frac{(k+1)k}{2}$ intersection points for $k + 1$ lines.
If an arrangement has $k + 1$ lines, there are N_{k+1} intersection points.
Imagine that k lines are blue and one is red. By the assumption, there must be $\frac{k(k-1)}{2}$ intersections between the k blue lines.
Since the red line is not parallel to any other, it must intersect each one of the k blue lines, and since no three lines have a common intersection point, each of these k intersections must be distinct from each other and from the any of the blue-blue intersection points.
Therefore, N_{k+1}

$$= \text{number of blue} \cdot \text{blue intersections} + \text{number of blue} \cdot \text{red intersections}$$
$$= N_k + k$$
$$= \frac{k(k-1)}{2} + k \text{(by assumption)}$$
$$= \frac{k}{2}(k - 1 + 2)$$
$$= \frac{k(k+1)}{2}$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Proposition 2: $R_n = \frac{n(n+1)}{2} + 1$

Base case $n = 1$:
The plane is divided into two regions by a single line; $R_1 = 2 = \frac{1(1+1)}{2} + 1$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for any valid arrangement of k lines for some $k \geq 1$

So $R_k = \frac{k(k+1)}{2} + 1$

Working towards: $\frac{(k+1)(k+2)}{2} + 1$ regions for $k + 1$ lines.

Imagine again a valid network of $k + 1$ lines, of which k are blue and one is red.
Without the red line there would be R_k regions.
The red line is separated into $k + 1$ parts by the k blue lines it crosses ($k - 1$ line segments that lie between two intersection points and the two rays that connect to the first and last intersection point of the line).
Each of these line segments must lie within a different region of the k line network, and so including the red line in the network creates an additional $k + 1$ regions.

Therefore,
$$\begin{aligned} R_{k+1} &= R_k + (k + 1) \\ &= \frac{k(k+1)}{2} + 1 + (k + 1) \text{(by assumption)} \\ &= \frac{1}{2}(k^2 + k + 2k + 2) + 1 \\ &= \frac{1}{2}(k^2 + 3k + 2) + 1 \\ &= \frac{1}{2}(k + 1)(k + 2) + 1 \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

6 Polynomials

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 6A

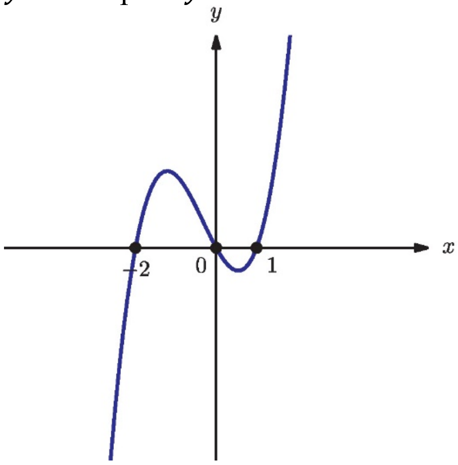
17

$y = 3x(x - 1)(x + 2)$

Coefficient of $x^3 > 0$ so positive cubic shape

x-intercepts at $x = 0, 1, -2$

y-intercept at $y = 0$



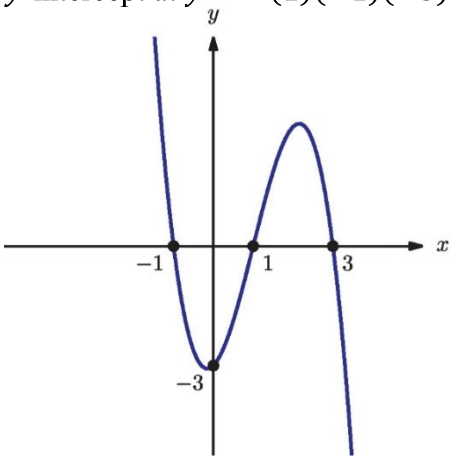
18

$y = -(x + 1)(x - 1)(x - 3)$

Coefficient of $x^3 < 0$ so negative cubic shape

x-intercepts at $x = -1, 1, 3$

y-intercept at $y = -(1)(-1)(-3) = -3$



19 **a**

$f(x) = 3x^3 - 12x$

$= 3x(x^2 - 4)$

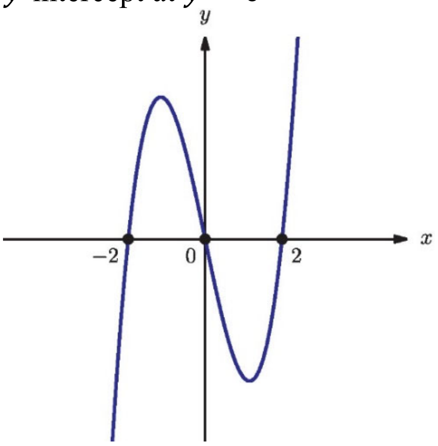
$= 3x(x - 2)(x + 2)$

b

Coefficient of $x^3 > 0$ so positive cubic shape

x-intercepts at $x = 0, 2, -2$

y-intercept at $y = 0$



20 **a**

$f(x) = 5x - 5x^3$

$= -5x(x^2 - 1)$

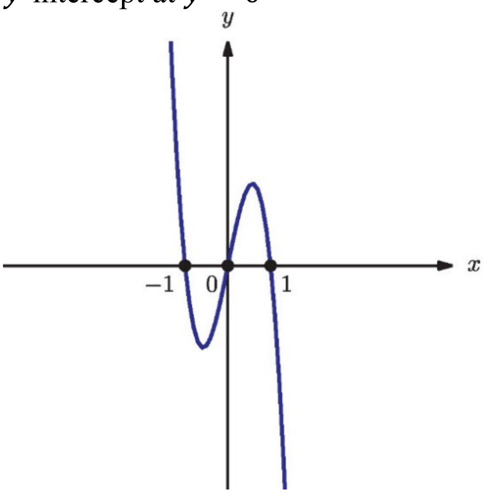
$= -5x(x - 1)(x + 1)$

b

Coefficient of $x^3 < 0$ so negative cubic shape

x-intercepts at $x = 0, 1, -1$

y-intercept at $y = 0$



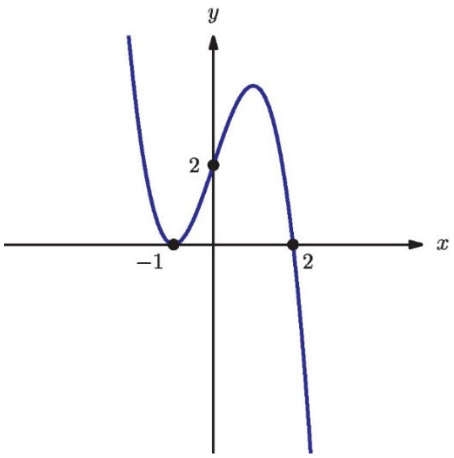
21 **a**

$y = (x + 1)^2(2 - x)$

Coefficient of $x^3 < 0$ so negative cubic shape

roots at $x = -1$ (repeated root), 2

y-intercept at $y = (1)^2(2) = 2$



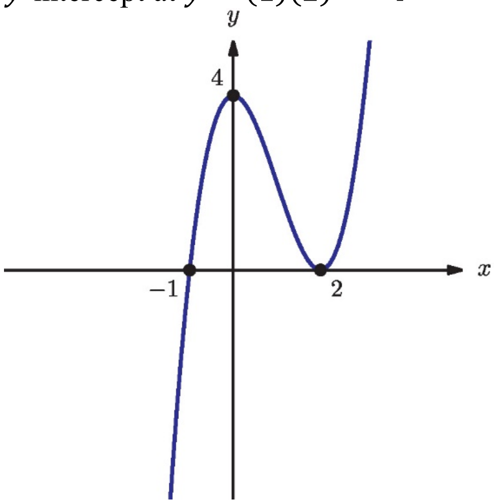
b

$y = (x+1)(2-x)^2$

Coefficient of $x^3 > 0$ so positive cubic shape

roots at $x = -1, 2$ (repeated root)

y-intercept at $y = (1)(2)^2 = 4$



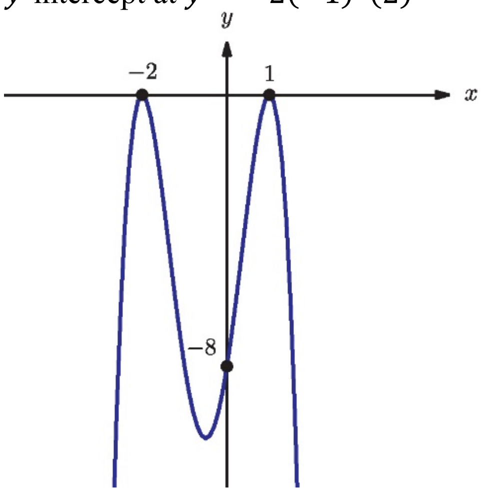
22

$y = -2(x-1)^2(x+2)^2$

Coefficient of $x^4 < 0$ so negative quartic shape

roots at $x = 1, -2$ (both repeated roots)

y-intercept at $y = -2(-1)^2(2)^2 = -8$



Worked solutions

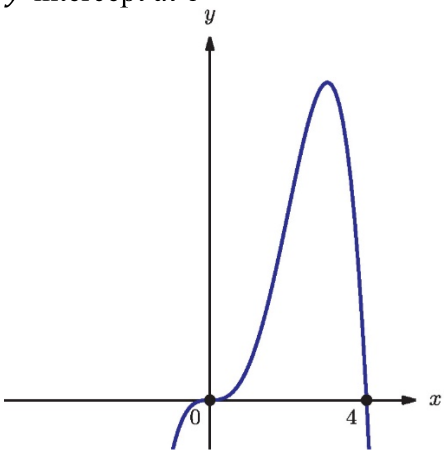
23 **a** $f(x) = 4x^3 - x^4 = x^3(4 - x)$

b

Coefficient of $x^4 < 0$ so negative quartic shape

roots at $x = 0$ (triple repeated root), 4

y-intercept at 0



24 **a**

roots at $x = -2, 3$ (repeated root)

$y = p(x + 2)(x - 3)^2 = p(x^3 - 4x^2 - 3x + 18)$

$y(0) = 36 = 18p \Rightarrow p = 2$

$y = 2x^3 - 8x^2 - 6x + 36$

$p = 2, q = -8, r = -6, s = 36$

b

roots at $x = 0$ (repeated root), 3

$y = px^2(x - 3) = p(x^3 - 3x^2)$

$y(2) = 4 = 4p(-1) \Rightarrow p = -1$

$y = -x^3 + 3x^2$

$p = -1, q = 3, r = \ s = 0$

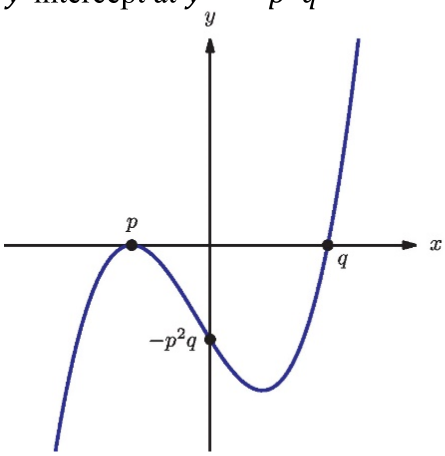
25 **a**

$y = (x - p)^2(x - q)$ where $p < q$

Coefficient of $x^3 > 0$ so positive cubic shape

roots at $x = p$ (repeated root), q

y-intercept at $y = -p^2q$



b There is only one solution to $y = k$ for positive values of k.

Exercise 6B

17 Let $f(x) = x^3 + ax + 7$
By the remainder theorem, $f(-2) = -5$
$-8 - 2a + 7 = -5$
$a = 2$

18 Let $f(x) = x^3 - 6x^2 + 4x + a$
By the remainder theorem, $f(3) = 2$
$27 - 54 + 12 + a = 2$
$a = 17$

19 Let $f(x) = x^2 + kx - 8k$
By the factor theorem, $f(k) = 0$
$k^2 + k^2 - 8k = 0$
$2k(k - 4) = 0$
$k = 0$ or 4

20 Let $f(x) = 6x^3 + ax^2 + bx + 8$
By the factor theorem, $f(-2) = 0$ and by the remainder theorem, $f(1) = -3$
$f(-2) = 0 = -48 + 4a - 2b + 8$
$2a - b = 20 \qquad (1)$
$f(1) = -3 = 6 + a + b + 8$
$a + b = -17 \qquad (2)$
$(1) + (2): 3a = 3$
$a = 1, b = -18$

21 Let $f(x) = x^3 + 8x^2 + ax + b$
By the factor theorem, $f(2) = 0$ and by the remainder theorem, $f(3) = 15$
$f(2) = 0 = 8 + 32 + 2a + b$
$2a + b = -40 \qquad (1)$
$f(3) = 15 = 27 + 72 + 3a + b$
$3a + b = -84 \qquad (2)$
$(2) - (1): a = -44$
$a = -44, b = 48$

22 Let $f(x) = x^2 - (k + 1)x - 3$
By the factor theorem, $f(k - 1) = 0$
$f(k - 1) = 0 = (k - 1)^2 - (k + 1)(k - 1) - 3$
$-2k - 1 = 0 \qquad (1)$
$k = -\frac{1}{2}$

23 $f(x) = x^3 - ax^2 - bx + 168 = (x - 3)(x - 7)(x - k)$
Comparing the constant coefficient: $168 = -21k \Rightarrow k = -8$
The three roots are therefore $3, 7, -8$

24 Let $f(x) = x^3 + ax^2 + 9x + b$
By the factor theorem, $f(11) = 0$ and by the remainder theorem, $f(-2) = -52$
$f(-2) = -52 = -8 + 4a - 18 + b$
$4a + b = -26$
$f(2) = 8 + 4a + 18 + b = 26 + 4a + b = 0$
By the factor (or remainder) theorem, it follows that the remainder when $f(x)$ is divided by $(x - 2)$ is 0.

Notice that we never needed the condition on $f(11)$ to answer the question, since a and b did not need to be evaluated.

25 **a** $f(4) = 64 - 2(16) - 11(4) + 12 = 0$ so by the factor theorem, $(x-4)$ is a factor of $f(x)$

b $x^3 - 2x^2 - 11x + 12 = (x-4)(x^2 + ax + b)$

Expanding and comparing coefficients:

$x^3: 1 = 1$

$x^2: -2 = a - 4 \Rightarrow a = 2$

$x^1: -11 = b - 4a \Rightarrow b = -3$

$x^0: 12 = -4b$ is consistent with $b = -3$

$x^3 - 2x^2 - 11x + 12 = (x-4)(x^2 + 2x - 3)$
$= (x-4)(x+3)(x-1)$

The solutions to $f(x) = 0$ are $x = 4, 1, -3$

26 **a**

$f(2) = 8 - 5(4) + 7(2) - 2 = 0$ so by the factor theorem, $(x-2)$ is a factor of $f(x)$

b $x^3 - 5x^2 + 7x - 2 = (x-2)(x^2 + ax + b)$

Expanding and comparing coefficients:

$x^3: 1 = 1$

$x^2: -5 = a - 2 \Rightarrow a = -3$

$x^1: 7 = b - 2a \Rightarrow b = 1$

$x^0: -2 = -2b$ is consistent with $b = 1$

$$x^3 - 5x^2 + 7x - 2 = (x-2)(x^2 - 3x + 1) = (x-2)\left(x - \frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{5}}{2}\right)$$

The solutions to $f(x) = 0$ are $x = 2, \frac{3\pm\sqrt{5}}{2}$

27 **a**

$f(x) = 6x^3 - x^2 + k$

By the factor theorem, $f\left(-\frac{1}{2}\right) = 0$

$$f\left(-\frac{1}{2}\right) = 0 = -\frac{6}{8} - \frac{1}{4} + k \Rightarrow k = 1$$

b

$6x^3 - x^2 + 1 = (2x+1)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 6 = 2a \Rightarrow a = 3$

$x^2: -1 = a + 2b \Rightarrow b = -2$

$x^1: 0 = b + 2c \Rightarrow c = 1$

$x^0: 1 = c$ is consistent with $c = 1$

$6x^3 - x^2 + 1 = (2x+1)(3x^2 - 2x + 1)$

The quadratic factor has discriminant $\Delta = (-2)^2 - 4(3)(1) = -8 < 0$

The quadratic therefore has no real roots, and so the only real root of the cubic $f(x)$ is at $x = -\frac{1}{2}$.

28 **a**

$f(x) = 2x^3 - 7x^2 - 3x + 3$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) - 7\left(\frac{1}{4}\right) - 3\left(\frac{1}{2}\right) + 3 = 0$$

By the factor theorem, since $f(x)$ has integer coefficients, $(2x-1)$ must be a factor of $f(x)$.

b

$2x^3 - 7x^2 - 3x + 3 = (2x - 1)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 2 = 2a \Rightarrow a = 1$

$x^2: -7 = 2b - a \Rightarrow b = -3$

$x^1: -3 = 2c - b \Rightarrow c = -3$

$x^0: 3 = -c$ is consistent with $c = -3$

$2x^3 - 7x^2 - 3x + 3 = (2x - 1)(x^2 - 3x - 3)$

The quadratic factor has discriminant $\Delta = (-3)^2 - 4(1)(-3) = 21 > 0$

The quadratic therefore has two distinct (non-integer) real roots and so the cubic has three distinct real roots.

29 **a**

$f(x) = 2x^3 - px^2 - 2p^2x + p^3$

$f(p) = 2p^3 - p^3 - 2p^3 + p^3 = 0$

By the factor theorem, $(x - p)$ is a factor of $f(x)$

b

$2x^3 - px^2 - 2p^2x + p^3 = (x - p)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 2 = a \Rightarrow a = 2$

$x^2: -p = b - pa \Rightarrow b = p$

$x^1: -2p^2 = c - pb \Rightarrow c = -p^2$

$x^0: p^3 = -pc$ is consistent with $c = -p^2$

$2x^3 - px^2 - 2p^2x + p^3 = (x - p)(2x^2 + px - p^2) = (x - p)(2x - p)(x + p)$

The roots of the function are $p, \frac{p}{2}$ and $-p$

30 **a**

$f(x) = x^3 - x^2 + k^2x - k^2$

$f(1) = 1 - 1 + k^2 - k^2 = 0$ so by the factor theorem, $(x - 1)$ is a factor of $f(x)$ for any value of k.

b

$x^3 - x^2 + k^2x - k^2 = (x - 1)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 1 = a \Rightarrow a = 1$

$x^2: -1 = b - a \Rightarrow b = 0$

$x^1: k^2 = c - b \Rightarrow c = k^2$

$x^0: -k^2 = -c$ is consistent with $c = k^2$

$x^3 - x^2 + k^2x - k^2 = (x - 1)(x^2 + k^2)$

The quadratic factor is a sum of squares and so has no real roots.

The only real root of the cubic is therefore $x = 1$.

31

$(ax + b)$ is a factor of $f(x) = ax^2 + abx + a^2$

By the factor theorem, $f\left(-\frac{b}{a}\right) = 0$

$$f\left(-\frac{b}{a}\right) = 0 = \frac{b^2}{a} - b^2 + a^2$$

$b^2(1 - a) = -a^3$

$$b = \pm\sqrt{\frac{a^3}{1 - a}}$$

32

$f(x) = 2x^4 - 3x^3 + 16x^2 - 27x - 18$ has factors $(2x+1)(x-2)$

$2x^4 - 3x^3 + 16x^2 - 27x - 18 = (2x+1)(x-2)(ax^2 + bx + c)$

$= (2x^2 - 3x - 2)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^4: 2 = 2a \Rightarrow a = 1$

$x^3: -3 = 2b - 3a \Rightarrow b = 0$

$x^2: 16 = 2c - 3b - 2a \Rightarrow c = 9$

$x^1: -27 = -3c - 2b$ is consistent with $b = 0, c = 9$

$x^0: -18 = -2c$ is consistent with $c = 9$

$2x^4 - 3x^3 + 16x^2 - 27x - 18 = (2x+1)(x-2)(x^2 + 9)$

$= (2x+1)(x-2)(x \pm 3\mathrm{i})$

The roots of $f(x) = 0$ are $x = -\frac{1}{2}, 2, \pm 3\mathrm{i}$

33

$f(x) = x^3 + ax^2 + 3x + b$

By the remainder theorem, $f(-1) = 6 = -1 + a - 3 + b$

$a + b = 10$

$f(1) = 1 + a + 3 + b = 4 + (a + b) = 14$

By the remainder theorem, the remainder when $f(x)$ is divided by $(x - 1)$ is $f(1) = 14$

34

$x^2 - 5x + 6 = (x-3)(x-2)$

Let $f(x) = 2x^3 - 15x^2 + ax + b$

By the factor theorem, if $(x-3)(x-2)$ is a factor of $f(x)$ then $f(3) = f(2) = 0$

$f(2) = 0 = 2(8) - 15(4) + 2a + b$

$2a + b = 44 \qquad (1)$

$f(3) = 0 = 2(27) - 15(9) + 3a + b$

$3a + b = 81 \qquad (2)$

$(2) - (1): a = 37$

$(2): b = 81 - 3a = -30$

35

Arithmetic sequence of three terms with central value 3 can be expressed as $3 - u, 3, 3 + u$

If these are the roots of the equation then

$x^3 + bx^2 + cx + d = (x - 3 + u)(x - 3)(x - 3 - u)$

$= (x-3)(x^2 - 6x + 9 - u^2)$

$= x^3 - 9x^2 + (27 - u^2)x + 3u^2 - 27$

$c = 27 - u^2, d = 3u^2 - 27$

$3c + d = 3(27 - u^2) + (3u^2 - 27) = 54$

Exercise 6C

21

Sum of the roots: $p + q = -\frac{1}{2}$

Product of the roots: $pq = \frac{3}{2}$

$(p-4)(q-4) = pq - 4(p+q) + 16 = \frac{3}{2} + 2 + 16 = \frac{39}{2}$

22

Sum of the roots: $p + q = \frac{a}{1} = a$

Product of the roots: $pq = \frac{3a}{1} = 3a$

a $5pq = 15a$

b $(p+q)^2 = a^2$

23

Sum of the roots: $\alpha + \beta = \frac{3}{k}$

Product of the roots: $\alpha\beta = \frac{5k}{k} = 5$

a $\alpha + \beta + 2 = 2 + \frac{3}{k}$

b $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{3}{5k}$

24

Sum of the roots: $p + q = \frac{-3}{a}$

Product of the roots: $pq = \frac{-a^2}{a} = -a$

$(p-q)^2 = p^2 + q^2 - 2pq = (p+q)^2 - 4pq = \frac{9}{a^2} + 4a$

25

Sum of the roots: $p + q = \frac{k}{1} = k$

Product of the roots: $pq = \frac{2k}{1} = 2k$

$\left(\frac{3}{p} + \frac{3}{q}\right) = \frac{3(p+q)}{pq} = \frac{3k}{2k} = \frac{3}{2}$

26

Sum of the roots: $p + q + r = \frac{-6}{3} = -2$

Product of the roots: $pqr = \frac{4}{3}$

$p^2qr + pq^2r + pqr^2 = (pqr)(p+q+r) = -\frac{8}{3}$

27

Sum of the roots: $\alpha+\beta+\gamma=\frac{5}{2}$

Product of the roots: $\alpha\beta\gamma=-\frac{3}{2}$

$$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}=-\frac{5}{3}$$

28

Sum of the roots: $p+q=\frac{3}{5}$

Product of the roots: $pq=\frac{2}{5}$

a $(p+3)(q+3)=pq+3(p+q)+9=\frac{56}{5}$

b For the new polynomial $5x^2+bx+c=0$:

Sum of the roots: $p+q+6=\frac{33}{5}=-\frac{b}{5}$

Product of the roots: $(p+3)(q+3)=\frac{56}{5}=\frac{c}{5}$

$b=-33, c=56$

29

This answer uses the fact that if a complex number is a root of a polynomial with real coefficients then its complex conjugate must also be a root. Students can use this result in examinations without proof.

a Since 3i and $3-\text{i}$ are roots, the other two roots are $-3\text{i}, 3+\text{i}$

b Sum of the roots: $3\text{i}+(-3\text{i})+(3-\text{i})+(3+\text{i})=\frac{a}{1}=a=6$

Product of the roots: $3\text{i}(-3\text{i})(3-\text{i})(3+\text{i})=\frac{d}{1}=d$

$d=9\times 10=90$

30 **a** Sum of the roots: $R_1+R_2=\frac{a}{3}$

Total resistance is $\frac{a}{3}$

b Product of the roots: $R_1R_2=\frac{2a}{3}$

Total resistance is $\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1}=\frac{R_1R_2}{R_1+R_2}=2$

31

Sum of the roots: $p+q+r+s=\frac{11}{1}=11$

The arithmetic mean of the roots is therefore $11\div 4=2.75$

32

Sum of the roots: $\alpha+\beta+\gamma=\frac{0}{3}=0\Rightarrow\alpha+\beta=-\gamma$

Full product of the roots: $\alpha\beta\gamma=\frac{-3}{3}=-1\Rightarrow\gamma=-\frac{1}{\alpha\beta}$

Substituting the second result into the first:

$\alpha+\beta=\frac{1}{\alpha\beta}$

33 **a**

Sum of the roots: $p + q = \frac{3}{5}$

Product of the roots: $pq = \frac{2}{5}$

$p^2 + q^2 = (p + q)^2 - 2pq = \frac{9}{25} - \frac{4}{5} = -\frac{11}{25}$

b

For the new quadratic $ax^2 + bx + c = 0$:

Sum of the roots: $-\frac{b}{a} = p^2 + q^2 = -\frac{11}{25}$

Product of the roots: $\frac{c}{a} = p^2q^2 = (pq)^2 = \frac{4}{25}$

For minimum integer coefficients select $a = 25$ so $b = 11, c = 4$

The quadratic is $25x^2 + 11x + 4 = 0$

34 **a**

Sum of the roots: $p + q = -\frac{1}{3}$

Product of the roots: $pq = -\frac{8}{3}$

$p^2 + q^2 = (p + q)^2 - 2pq = \frac{1}{9} + \frac{16}{3} = \frac{49}{9}$

b

For the new quadratic $ax^2 + bx + c = 0$:

Sum of the roots: $-\frac{b}{a} = p^2 + q^2 = \frac{49}{9}$

Product of the roots: $\frac{c}{a} = p^2q^2 = (pq)^2 = \frac{64}{9}$

For minimum integer coefficients select $a = 9$ so $b = -49, c = 64$

The quadratic is $9x^2 - 49x + 64 = 0$

35 **a** Using binomial expansion,

$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 = p^3 + q^3 + 3pq(p + q)$

b

Sum of the roots: $p + q = \frac{1}{4}$

Product of the roots: $pq = \frac{2}{4} = \frac{1}{2}$

From part **a**, $p^3 + q^3 = (p + q)^3 - 3pq(p + q)$

$p^3 + q^3 = \left(\frac{1}{4}\right)^3 - 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = -\frac{23}{64}$

c

For the new quadratic $ax^2 + bx + c = 0$:

Sum of the roots: $-\frac{b}{a} = p^3 + q^3 = -\frac{23}{64}$

Product of the roots: $\frac{c}{a} = p^3q^3 = (pq)^3 = \frac{1}{8}$

For minimum integer coefficients select $a = 64$ so $b = 23, c = 8$

The quadratic is $64x^2 + 23x + 8 = 0$

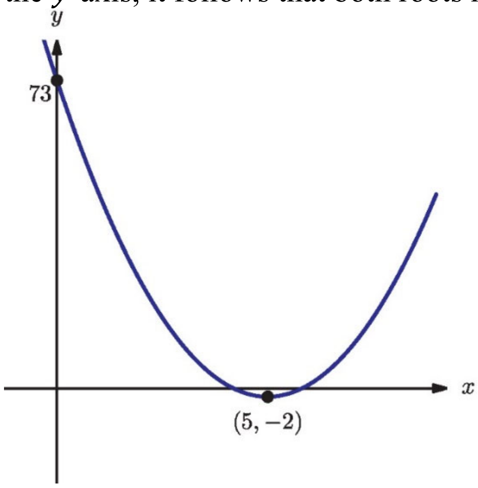
36 **a** $\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2=\alpha+\beta+2\sqrt{\alpha\beta}$

b

$$3x^2-30x+73=3(x^2-10x)+73$$
$$=3((x-5)^2-25)+73$$
$$=3(x-5)^2-2$$

c

From part **b**, the vertex of the parabola is at $(5,-2)$.
From the equation, the y-intercept is at $(0,73)$
Since the curve is a positive quadratic, the vertex is a minimum; since it lies midway between the two roots, and the curve has passed into the positive region to the right of the y-axis, it follows that both roots must be real and positive.



d

Sum of the roots: $\alpha+\beta=10$

Product of the roots: $\alpha\beta=\frac{73}{3}$

From part **a**: $\sqrt{\alpha}+\sqrt{\beta}=\sqrt{\alpha+\beta+2\sqrt{\alpha\beta}}=\sqrt{10+2\sqrt{\frac{73}{3}}}$

37

Sum of the roots: $p+2p+3p+4p=-\frac{b}{1}\Rightarrow b=-10p$

Product of the roots: $p(2p)(3p)(4p)=\frac{e}{1}\Rightarrow e=24p^4$

Then $3b^4=30000p^4=1250(24p^4)=1250e$

38

Sum of the roots: $\alpha+\beta=-\frac{2}{5}$

Product of the roots: $\alpha\beta=\frac{3}{5}$

For the new quadratic $ax^2+bx+c=0$:

Sum of the roots: $-\frac{b}{a}=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=-\frac{26}{15}$

Product of the roots: $\frac{c}{a}=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$

For minimum integer coefficients select $a=15$ so $b=26, c=15$
The quadratic is $15x^2+26x+15=0$

39 **a**

$(x-p)(x-q)(x-r) = (x^2-(p+q)x+pq)(x-r)$
$= x^3-(p+q+r)x^2+(pq+qr+rp)x-pqr$

Comparing the coefficient in x^1, $pq+qr+rp = 3a^2$

b

Comparing the coefficient in x^2, $p+q+r = -2a$

$(p+q+r)^2 = p^2+q^2+r^2+2(pq+qr+rp)$

Therefore $p^2+q^2+r^2 = (p+q+r)^2-2(pq+qr+rp) = 4a^2-6a^2 = -2a^2$

c

The sum of the squares of the roots gives a negative value; this is not possible if all the roots have real values, so we conclude that they are not all real.

40 **ai**

$(p+q+r)^2 = p(p+q+r)+q(p+q+r)+r(p+q+r)$
$= p^2+pq+pr+qp+q^2+qr+rp+rq+r^2$
$= p^2+q^2+r^2+2(pq+qr+rp)$

Therefore $p^2+q^2+r^2 = (p+q+r)^2-2(pq+qr+rp)$

aii

$(pq+qr+rp)^2 = pq(pq+qr+rp)+qr(pq+qr+rp)+rp(pq+qr+rp)$
$= p^2q^2+pq^2r+p^2qr+pq^2r+q^2r^2+pqr^2+p^2qr+pqr^2$
$+p^2r^2$
$= p^2q^2+q^2r^2+r^2p^2+2pqr(p+q+r)$

Therefore $p^2q^2+q^2r^2+r^2p^2 = (pq+qr+rp)^2-2pqr(p+q+r)$

bi

Sum of the roots: $p+q+r = -\frac{b}{a}$

Product of the roots: $pqr = -\frac{d}{a}$

bii

$a(x-p)(x-q)(x-r) = a(x^2-(p+q)x+pq)(x-r)$
$= ax^3-a(p+q+r)x^2+a(pq+qr+rp)x-apqr$

Comparing the coefficient in x^1, $pq+qr+rp = \frac{c}{a}$

ci

Sum of the roots: $\alpha+\beta+\gamma = \frac{0}{2} = 0$

Paired products of the roots: $\alpha\beta+\beta\gamma+\gamma\alpha = -\frac{7}{2}$

Full product of the roots: $\alpha\beta\gamma = \frac{-4}{2} = -2$

Using part **ai**: $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha) = 7$

cii

$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma) = \frac{49}{4}$

$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 4$

ciii

For the new cubic $ax^3+bx^2+cx+d = 0$:

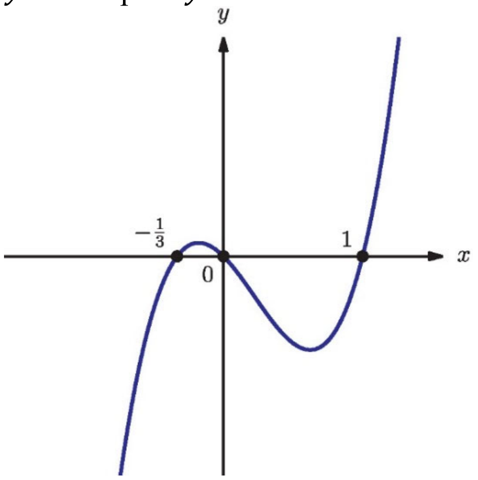
Sum of the roots: $-\frac{b}{a} = \alpha^2+\beta^2+\gamma^2 = 7$

Paired products of the roots: $\frac{c}{a} = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = \frac{49}{4}$

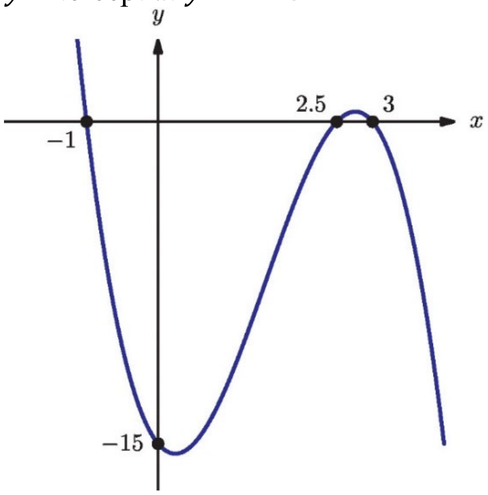
Full product of the roots: $-\frac{d}{a} = \alpha^2\beta^2\gamma^2 = 4$
For minimum integer coefficients select $a = 4$ so $b = -28, c = 49, d = -16$
The cubic is $4x^3 - 28x^2 + 49x - 16 = 0$

Mixed Practice

1 **a**
$f(x) = 3x^3 - 2x^2 - x$
$= x(3x^2 - 2x - 1)$
$= x(3x + 1)(x - 1)$
b
Coefficient of $x^3 > 0$ so positive cubic shape
x-intercepts at $x = 0, -\frac{1}{3}, 1$
y-intercept at $y = 0$



2
$y = (x - 3)(x + 1)(5 - 2x)$
Coefficient of $x^3 < 0$ so negative cubic shape
x-intercepts at $x = -1, \frac{5}{2}, 3$
y-intercept at $y = -15$



3 roots at $x = -2, -1, 1$
$y = a(x+2)(x+1)(x-1) = a(x+2)(x^2-1) = a(x^3+2x^2-x-2)$
$y(0) = -6 = -2a \Rightarrow a = 3$
$y = 3x^3 + 6x^2 - 3x - 6$

4 $y = 0.3(x-p)(x^2+k^2)$
root at $x = 2$ so $p = 2$
$y(0) = -5.4 = -0.3pk^2 \Rightarrow k^2 = 9$
$k = 3$

5 $\mathrm{f}(x) = (ax+b)^3$
By the remainder theorem, $\mathrm{f}(2) = 8$ and $\mathrm{f}(-3) = -27$
$\mathrm{f}(2) = 8 = (2a+b)^3 \Rightarrow 2a + b = 2$ (1)
$\mathrm{f}(-3) = -27 = (-3a+b)^3 \Rightarrow -3a + b = -3$ (2)
$(1) - (2): 5a = 5 \Rightarrow a = 1, b = 0$

6 $\mathrm{f}(x) = x^3 + 4x^2 + ax + b$
By the factor theorem, $f(1) = 0$ and by the remainder theorem, $\mathrm{f}(2) = 17$
$\mathrm{f}(1) = 0 = 1 + 4 + a + b \Rightarrow a + b = -5$ (1)
$\mathrm{f}(2) = 17 = 8 + 16 + 2a + b \Rightarrow 2a + b = -7$ (2)
$(2) - (1): a = -2 \Rightarrow a = -2, b = -3$

7 $\mathrm{f}(x) = x^4 + px^2 - x + q$
By the factor theorem, $f(-1) = 0$ and by the remainder theorem, $\mathrm{f}(3) = 52$
$\mathrm{f}(-1) = 0 = 1 + p + 1 + q \Rightarrow p + q = -2$ (1)
$\mathrm{f}(3) = 52 = 81 + 9p - 3 + q \Rightarrow 9p + q = -26$ (2)
$(2) - (1): 8p = -24 \Rightarrow p = -3, q = 1$

8 Real coefficients: Complex roots occur in conjugate pairs
$5x^2 + bx + c = 0$ has roots $4 + 7\mathrm{i}$ and $4 - 7\mathrm{i}$

Sum of the roots: $-\frac{b}{5} = 8 \Rightarrow b = -40$

Product of the roots: $\frac{c}{5} = (4+7\mathrm{i})(4-7\mathrm{i}) = 65 \Rightarrow c = 325$

9

Sum of the roots: $-\frac{-a}{4} = \frac{3}{2} \Rightarrow a = 6$

10

Sum of the roots: $p + q + r + s = \frac{-8}{k}$

Mean of the roots: $\frac{p+q+r+s}{4} = -\frac{1}{2}$

$-\frac{8}{k} = -2 \Rightarrow k = 4$

11

Product of the roots: $a(2a)(3a)(4a) = \frac{1536}{4} = 384$

$24a^4 = 384$
$a^4 = 16$
$a = \pm 2$

12 **a**

Product of the roots: $abc = -\frac{5}{3}$

b

Sum of the roots: $a + b + c = -\frac{2}{3}$

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{c + b + a}{abc} = \frac{2}{5}$$

13

Let $\mathrm{f}(x) = 2x^3 + kx^2 + 6x + 32$ and $\mathrm{g}(x) = x^4 - 6x^2 - k^2x + 9$

By the remainder theorem, $\mathrm{f}(-1) = \mathrm{g}(-1)$

$\mathrm{f}(-1) = -2 + k - 6 + 32 = k + 24$

$\mathrm{g}(-1) = 1 - 6 + k^2 + 9 = 4 + k^2$

$4 + k^2 = k + 24$

$k^2 - k - 20 = 0$

$(k - 5)(k + 4) = 0$

$k = 5$ or -4

14

With four distinct roots, the problem could be approached by forming five simultaneous equations in a, b, c, d, e by evaluating the function at each root and at $x = 0$, then using the calculator to solve the system of equations. With a repeated root, the constructive solution below is arguably simpler. However, a system of simultaneous equations could be set up using the three roots, $x = 0$ and the fact that $y'(-3) = 0$ as the fifth equation. This approach is given below as an alternative; it avoids some of the complexity of expanding multiple brackets. Either way, the final answer can readily be checked by plotting the proposed curve on the GDC.

Roots at $x = -3$ (repeated root), 1, 3

$$\begin{aligned} y &= a(x+3)^2(x-1)(x-3) \\ &= a(x^2 + 6x + 9)(x^2 - 4x + 3) \\ &= a(x^4 + 2x^3 - 12x^2 - 18x + 27) \end{aligned}$$

$y(0) = 27 = 27a \Rightarrow a = 1$

$y = x^4 + 2x^3 - 12x^2 - 18x + 27$

$a = 1, b = 2, c = -12, d = -18, e = 27$

Alternatively:

$$\begin{cases} y(-3) = 0: & 81a - 27b + 9c - 3d + e = 0 & (1) \\ y(0) = 27: & 0a + 0b + 0c + 0d + e = 27 & (2) \\ y(1) = 0: & a + b + c + d + e = 0 & (3) \\ y(3) = 0: & 81a + 27b + 9c + 3d + e = 0 & (4) \\ y'(-3) = 0 & -108a + 27b - 6c + d = 0 & (5) \end{cases}$$

Solving this system using GDC:

$a = 1, b = 2, c = -12, d = -18, e = 27$

15

Roots at $x = -1,\ 1$ (triple repeated root)

$$\begin{aligned} y &= a(x+1)(x-1)^3 \\ &= a(x^2 - 1)(x^2 - 2x + 1) \\ &= a(x^4 - 2x^3 + 2x - 1) \end{aligned}$$

$y(0) = 1 = -a \Rightarrow a = -1$

$y = -x^4 + 2x^3 - 2x + 1$

16

Roots at $x = 0$ (triple repeated root), 12

$y = ax^3(x - 12)$

$= a(x^4 - 12x^3)$

$y(9) = -27 = -2187a \Rightarrow a = \frac{1}{81}$

$y = \frac{1}{81}x^4 - \frac{4}{27}x^3$

17 **a**

$f(x) = 2x^3 - 5x^2 + x + 2$

$f(2) = 2(8) - 5(4) + 2 + 2 = 0$

By the factor theorem, $(x - 2)$ must be a factor of $f(x)$

b

$f(x) = 2x^3 - 5x^2 + x + 2 = (x - 2)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 2 = a \Rightarrow a = 2$

$x^2: -5 = b - 2a \Rightarrow b = -1$

$x^1: 1 = c - 2b \Rightarrow c = -1$

$x^0: 2 = -2c$ is consistent with $c = -1$

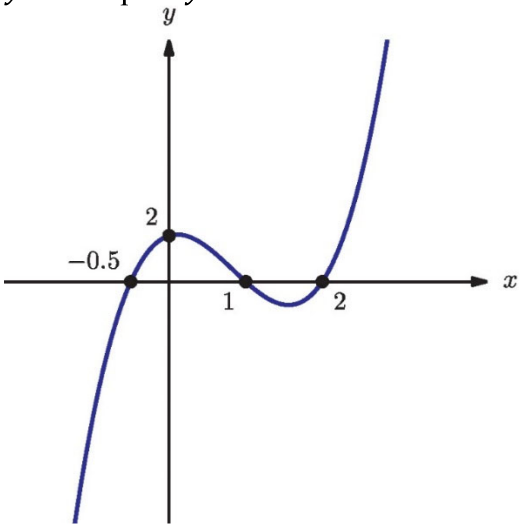
$f(x) = (x - 2)(2x^2 - x - 1) = (x - 2)(2x + 1)(x - 1)$

c

Coefficient of $x^3 > 0$ so positive cubic shape

x-intercepts at $x = 2, -0.5, 1$

y-intercept at $y = 2$



18 **a**

$f(x) = x^3 - 4x^2 + x + 6$

$f(-1) = -1 - 4(1) + (-1) + 6 = 0$

By the factor theorem, $(x + 1)$ must be a factor of $f(x)$

b

$f(x) = x^3 - 4x^2 + x + 6 = (x + 1)(ax^2 + bx + c)$

Expanding and comparing coefficients:

$x^3: 1 = a \Rightarrow a = 1$

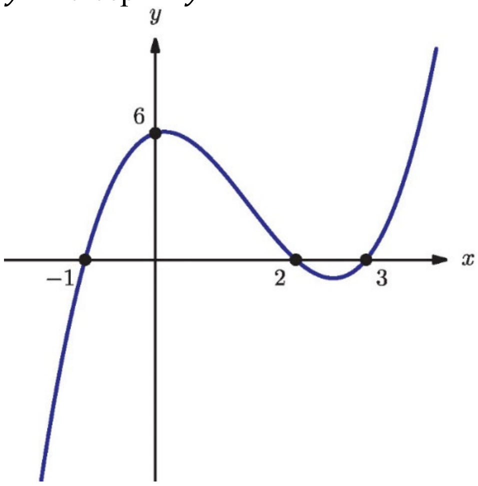
$x^2: -4 = b + a \Rightarrow b = -5$

$x^1: 1 = c + b \Rightarrow c = 6$

$x^0: 6 = c$ is consistent with $c = 6$
$f(x) = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$

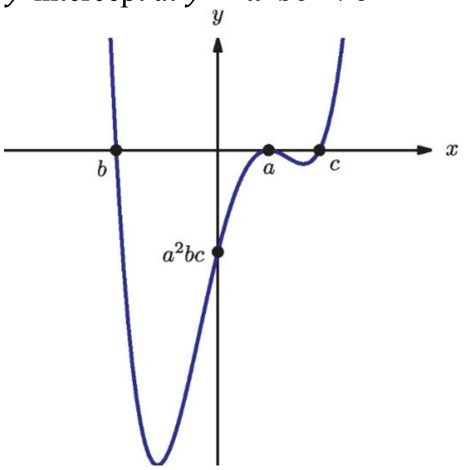
c

Coefficient of $x^3 > 0$ so positive cubic shape
x-intercepts at $x = -1, 2, 3$
y-intercept at $y = 6$



19

$y = (x - a)^2(x - b)(x - c)$ where $b < 0 < a < c$
Coefficient of $x^4 > 0$ so positive quartic shape
x-intercepts at $x = b, a$ (repeated root), c
y-intercept at $y = a^2bc < 0$



20

Let $f(x) = x^3 + ax^2 - 7x + 15$
$f(-3) = 0 = -27 + 9a + 21 + 15 \Rightarrow a = -1$
By the factor theorem, if $f(-3) = 0$ then $(x + 3)$ is a factor of $f(x)$
$f(x) = x^3 - x^2 - 7x + 15 = (x + 3)(ax^2 + bx + c)$
Expanding and comparing coefficients:
$x^3: 1 = a \Rightarrow a = 1$
$x^2: -1 = b + 3a \Rightarrow b = -4$
$x^1: -7 = c + 3b \Rightarrow c = 5$
$x^0: 15 = 3c$ is consistent with $c = 5$
$f(x) = (x + 3)(x^2 - 4x + 5) = (x + 3)(x - 2 + i)(x - 2 - i)$
The roots are $-3, 2 \pm i$

21

Let $\mathrm{f}(x) = ax^3 + bx^2 + 177x - 210$
$\mathrm{f}(2) = 0 = 8a + 4b + 354 - 210$
$8a + 4b = -144$
$2a + b = -36 \quad (1)$
Sum of the roots: $2 + 12 = -\frac{b}{a}$
$b = -14a \quad (2)$
Substituting (2) into (1):
$-12a = -36$
$a = 3, b = -42$

22

Let $\mathrm{f}(x) = 3x^3 - 12x^2 + 16x - 8$
$\mathrm{f}(0) = -8$
$\mathrm{f}(1) = 3 - 12 + 16 - 8 = -1$
$\mathrm{f}(2) = 3(8) - 12(4) + 16(2) - 8 = 0$
By the factor theorem, $(x - 2)$ must be a factor of $\mathrm{f}(x)$
$\mathrm{f}(x) = 3x^3 - 12x^2 + 16x - 8 = (x - 2)(ax^2 + bx + c)$
Expanding and comparing coefficients:
$x^3: 3 = a \Rightarrow a = 3$
$x^2: -12 = b - 2a \Rightarrow b = -6$
$x^1: 16 = c - 2b \Rightarrow c = 4$
$x^0: -8 = -2c$ is consistent with $c = 4$
$\mathrm{f}(x) = (x - 2)(3x^2 - 6x + 4)$
The roots of the quadratic factor are

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(4)}}{6} = 1 \pm \frac{\sqrt{-12}}{6} = 1 \pm \frac{\sqrt{3}}{3}\mathrm{i}$$

23

Sum of the roots: $p + q = -\frac{b}{a}$
Product of the roots: $pq = \frac{c}{a}$
$\frac{1}{p} + \frac{1}{q} = \frac{p + q}{pq} = -\frac{b}{c} = 3$
$-b = 3c$
$b + 3c = 0$

24

$\mathrm{f}(x) = (ax + b)^4$
By the remainder theorem, $\mathrm{f}(2) = 16, \mathrm{f}(-1) = 81$
$\mathrm{f}(2) = (2a + b)^4 = 16 \Rightarrow 2a + b = \pm 2 \quad (1)$
$\mathrm{f}(-1) = (-a + b)^4 = 81 \Rightarrow -a + b = \pm 3 \quad (2)$
Taking each combination of simultaneous equations and solving
$(1) - (2)$: $3a = \pm 1$ or ± 5

	$b - a = 3$	$b - a = -3$
$2a + b = 2$	$a = -\frac{1}{3}, b = \frac{8}{3}$	$a = \frac{5}{3}, b = -\frac{4}{3}$
$2a + b = -2$	$a = -\frac{5}{3}, b = \frac{4}{3}$	$a = \frac{1}{3}, b = -\frac{8}{3}$

25

$f(x) = x^4 + px^3 + 14x^2 - 18x + q$

Real coefficients: Complex roots occur in conjugate pairs

3i and $1 - 2i$ are roots and therefore so are $-3i$ and $1 + 2i$

$f(x) = (x - 3i)(x + 3i)(x - 1 + 2i)(x - 1 - 2i)$

$= (x^2 + 9)(x^2 - 2x + 5)$

$= x^4 - 2x^3 + 14x^2 - 18x + 45$

$p = -2, q = 45$

26 **a**

Sum of the roots: $p + q = \frac{4}{3}$

Product of the roots: $pq = \frac{7}{3}$

$p^2 + q^2 = (p + q)^2 - 2pq = \frac{16}{9} - \frac{14}{3} = -\frac{26}{9}$

b

For the new quadratic $ax^2 + bx + c = 0$:

Sum of the roots: $-\frac{b}{a} = p^2 + q^2 = -\frac{26}{9}$

Product of the roots: $\frac{c}{a} = p^2q^2 = \left(\frac{7}{3}\right)^2 = \frac{49}{9}$

For minimum integer coefficients select $a = 9$ so $b = 26, c = 49$

The quadratic is $9x^2 + 26x + 49 = 0$

27 **a**

Sum of the roots: $a + b + c + d + e = -\frac{(-6)}{3} = 2$

b

Translation of the original graph 4 units to the right, so all roots will increase by 4

The sum of the new roots is therefore $2 + 5 \times 4 = 22$

28 **a**

Sum of the roots: $a + b + c + d = -\frac{2}{5}$

Product of the roots: $abcd = \frac{3}{5}$

b

Horizontal stretch with scale factor $\frac{1}{3}$ so all new roots will have values $\frac{1}{3}$ of the original roots.

The product of the new roots is therefore $\frac{3}{5} \times \left(\frac{1}{3}\right)^4 = \frac{1}{135}$

29 **a**

$f(x) = 4x^3 + 2ax - 7a$

By the remainder theorem, $f(a) = -10$

$f(a) = -10 = 4a^3 + 2a^2 - 7a \Rightarrow 4a^3 + 2a^2 - 7a + 10 = 0$

From GDC, this cubic has a single real root $a = -2$

b

$f(x) = 4x^3 - 4x + 14$

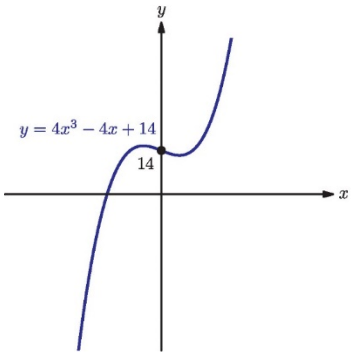
This cubic has turning points where $f'(x) = 0$

$f'(x) = 12x^2 - 4 = 4(3x^2 - 1)$

Turning points are at $x = \pm\frac{1}{\sqrt{3}}$

$$\mathrm{f}\left(\frac{1}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} - \frac{4}{\sqrt{3}} + 14 > 0$$
$$\mathrm{f}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{4}{3\sqrt{3}} + \frac{4}{\sqrt{3}} + 14 > 0$$
Since both turning points lie above the x-axis, there must only be a single real root to the equation $\mathrm{f}(x) = 0$.

A coherent argument such as the one above is acceptable; alternatively, students could offer a well-labelled graph from the GDC indicating the single root.



30
Let $\mathrm{f}(x) = 5x^3 + 48x^2 + 100x + 2 - a$

Sum of the roots: $r_1 + r_2 + r_3 = -\frac{48}{5}$

Product of the roots: $r_1r_2r_3 = -\frac{2-a}{5} = \frac{a-2}{5}$

If $r_1 + r_2 + r_3 + r_1r_2r_3 = 0$ then $-\frac{48}{5} + \frac{a-2}{5} = 0$

$a = 50$

31 **a** $x^2 - 4x + 5 = 0$

Using quadratic formula:

$$x = \frac{4 + \sqrt{(-4)^2 - 4(1)(5)}}{2} = 2 \pm \mathrm{i}$$

b Let $\mathrm{f}(x) = x^4 - 4x^3 + 8x^2 + ax + b = (x^2 - 4x + 5)(x^2 + px + q)$

Expanding and comparing coefficients:

$x^4: 1 = 1$

$x^3: -4 = p - 4 \Rightarrow p = 0$

$x^2: 8 = q - 4p + 5 \Rightarrow q = 3$

$x^1: a = 5p - 4q = -12$

$x^0: b = 5q = 15$

$a = -12, b = 15$

32
$x^2 - 4x + 3 = (x-1)(x-3)$

$(x-1)$ and $(x-3)$ are factors of $\mathrm{f}(x) = x^3 + ax^2 + 27x + b$

By the factor theorem, $\mathrm{f}(1) = \mathrm{f}(3) = 0$

$\mathrm{f}(1) = 1 + a + 27 + b = 0 \Rightarrow a + b = -28$ (1)

$\mathrm{f}(3) = 27 + 9a + 81 + b = 0 \Rightarrow 9a + b = -108$ (2)

$(2) - (1): 8a = -80$

$a = -10, b = -18$

33 **a**

$\mathrm{f}(x) = (x-a)^2\,\mathrm{g}(x)$ for some polynomial $\mathrm{g}(x)$

By the product rule:

$\mathrm{f}'(x) = 2(x-a)\mathrm{g}(x) + (x-a)^2\mathrm{g}'(x) = (x-a)[2\mathrm{g}(x) + (x-a)\mathrm{g}'(x)]$

Since $\mathrm{g}(x)$ is a polynomial, $\mathrm{g}'(x)$ must also be a polynomial, so $(x-a)$ is a factor of $\mathrm{f}'(x)$

b

Let $\mathrm{f}(x) = 2x^4 + bx^3 + 11x^2 - 12x + e$ so $\mathrm{f}'(x) = 8x^3 + 3bx^2 + 22x - 12$

By the factor theorem, $\mathrm{f}(2) = 0$ and by the reasoning in part **a**, $\mathrm{f}'(2) = 0$

$\mathrm{f}(2) = 0 = 2(16) + 8b + 11(4) - 12(2) + e \Rightarrow 8b + e = -52$ (1)

$\mathrm{f}'(2) = 0 = 8(8) + 3b(4) + 22(2) - 12 \Rightarrow 12b = -96$ (2)

$(2): b = -8$

$(1): e = -52 - 8b = 12$

34

Let $\mathrm{f}(x) = 6x^3 - 19x^2 + cx + d$

Three roots forming a geometric sequence with second term 1 can be expressed as r^{-1}, 1 and r

Product of the roots: $r^{-1} \times 1 \times r = -\dfrac{d}{6} \Rightarrow d = -6$

Also, $\mathrm{f}(1) = 0 = 6 - 19 + c + d \Rightarrow c = 13 - d = 19$

35 **a**

Let $\mathrm{f}(x) = x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$

$$(x-\alpha)(x-\beta)(x-\gamma) = (x^2 - (\alpha+\beta)x + \alpha\beta)(x-\gamma)$$
$$= x^3 - (\alpha+\beta+\gamma)x^2 + \big(\alpha\beta + \gamma(\alpha+\beta)\big)x - \alpha\beta\gamma$$

Comparing coefficients:

ai $x^2: p = -(\alpha+\beta+\gamma)$

aii $x^1: q = \alpha\beta + \gamma(\alpha+\beta) = \alpha\beta + \beta\gamma + \gamma\alpha$

aiii $x^0: c = -\alpha\beta\gamma$

b $\mathrm{f}(x) = x^3 - 6x^2 + 18x + c$

bi If the roots form an arithmetic sequence then they have values $a-d, a, a+d$

Sum of the roots: $(a-d) + a + (a+d) = -p = 6 \Rightarrow 3a = 6$

One of the roots has value $a = 2$

bii By the factor theorem, if 2 is a root then $\mathrm{f}(2) = 0$

$\mathrm{f}(2) = 0 = 8 - 6(4) + 18(2) + c = 20 + c \Rightarrow c = -20$

Alternatively: $q = 18 = (2-d)2 + 2(2+d) + (2+d)(2-d)$
$= 12 - d^2$

$d = \mathrm{i}\sqrt{6}$

$c = -2\big(2 + \mathrm{i}\sqrt{6}\big)\big(2 - \mathrm{i}\sqrt{6}\big) = -2(4+6) = -20$

c If the roots form a geometric sequence then they have values br^{-1}, b, br

Sum of the roots: $b(r^{-1} + 1 + r) = 6$ (1)

Paired product of the roots: $br^{-1}b + bbr + brbr^{-1} = 18$

$b^2(r^{-1} + r + 1) = 18$ (2)

$(2) \div (1): b = 3$

Product of the roots: $c = -br^{-1} \times b \times br = -b^3 = -27$
$c = -27$

7 Functions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 7A

9 **a**

Vertical asymptotes occur at the roots of the denominator

$4x^2 = 9$

$x = \pm\frac{3}{2}$

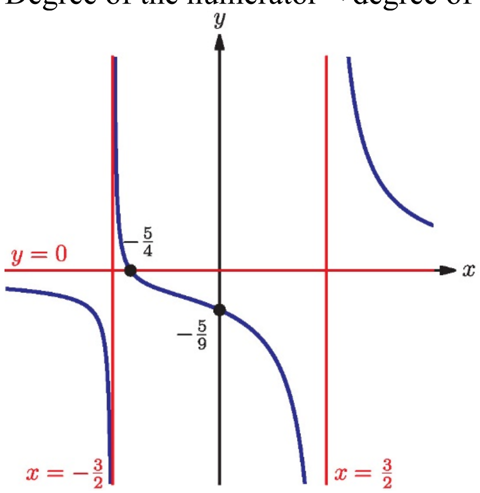
b

When $x = 0, y = -\frac{5}{9}$

x-intercept at root of numerator:

$4x + 5 = 0 \Rightarrow \left(-\frac{5}{4}, 0\right)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



10 **a**

Vertical asymptotes occur at the roots of the denominator

$3x^2 + 2x - 8 = 0$

$(3x - 4)(x + 2) = 0$

$x = \frac{4}{3}$ or -2

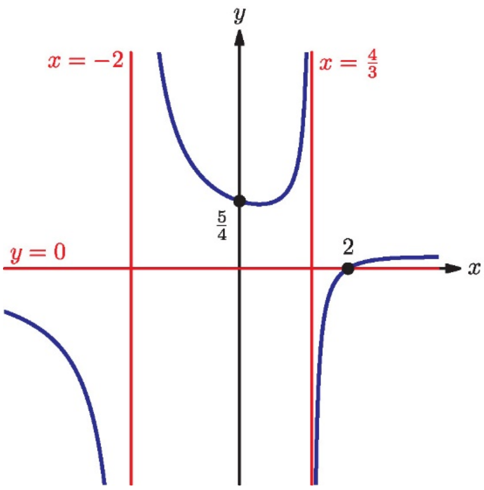
b

When $x = 0, y = \frac{10}{8} = \frac{5}{4}$

x-intercept at root of numerator:

$5x - 10 = 0 \Rightarrow (2,0)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



11 **a**

Require $x = -4$ to be a root of the denominator. Substituting:

$2(-4)^2 + k(-4) - 12 = 0$

$20 - 4k = 0$

$k = 5$

b

$2x^2 + 5x - 12 = 0$

$(x + 4)(2x - 3) = 0$

$x = -4 \text{ or } x = \frac{3}{2}$

The other vertical asymptote is $x = \frac{3}{2}$

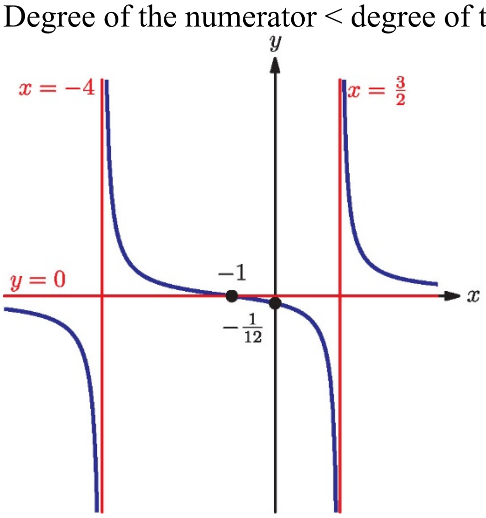
c

When $x = 0, y = -\frac{1}{12}$

x-intercept at root of numerator:

$x + 1 = 0 \Rightarrow (-1,0)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



12 **a**

For $y = \dfrac{3x}{x^2 - 2x + 1}$:

Vertical asymptotes at roots of denominator:

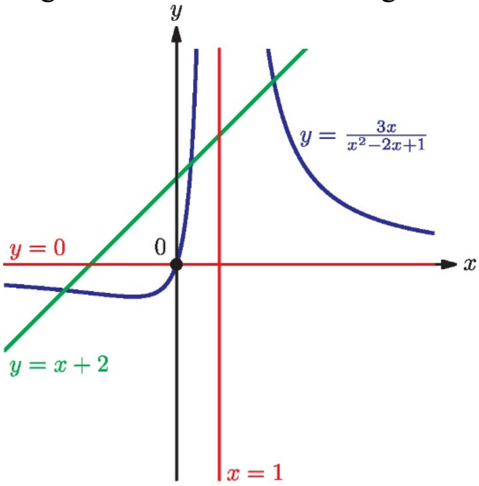
$(x-1)^2 = 0 \Rightarrow x = 1$ (multiplicity 2)

Axis intercept:

When $x = 0, y = 0$

x-intercept at root of numerator: $(0, 0)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



b

The line crosses the curve of the rational function in three places, so there are three solutions.

13 **a**

For $y = \dfrac{x-1}{2x^2 + 5x - 3}$:

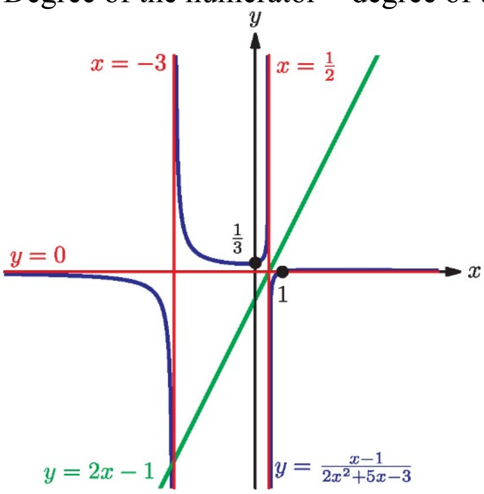
Vertical asymptotes at roots of denominator: $(2x-1)(x+3) = 0 \Rightarrow x = \frac{1}{2}$ or -3

Axis intercepts:

When $x = 0, y = \dfrac{1}{3}$

x-intercept at root of numerator: $(1, 0)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



b

The line crosses the curve of the rational function in one place, so there is one solution.

14 **ai**

$\frac{2x-3}{x^2+4} = k$

$kx^2 - 2x + 3 + 4k = 0$

Real solution when discriminant $\Delta \geq 0$

$\Delta = 4 - 4(3+4k)(k) \geq 0$

$16k^2 + 12k - 4 \leq 0$

$4k^2 + 3k - 1 \leq 0$

aii

Factorising:

$(4k-1)(k+1) \leq 0$

The curve intersects $y = k$ for $-1 \leq k \leq \frac{1}{4}$, so the turning points are at $y = -1$ and $y = \frac{1}{4}$

$k = 1{:}\ -x^2 - 2x - 1 = 0 \Rightarrow (x+1)^2 = 0$. The turning point is $(-1, -1)$

$k = \frac{1}{4}: \frac{1}{4}x^2 - 2x + 4 = 0 \Rightarrow (x-4)^2 = 0$. The turning point is $\left(4, \frac{1}{4}\right)$

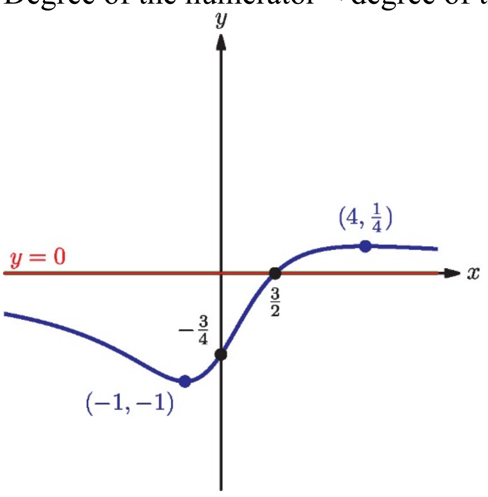
b

When $x = 0, y = -\frac{3}{4}$

x-intercept at root of numerator:

$2x - 3 = 0 \Rightarrow \left(\frac{3}{2}, 0\right)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



15 **ai**

If there are real roots the discriminant $\Delta \geq 0$

$\Delta = \left(-(k+1)\right)^2 - 4(k)(-2k-2) = 9k^2 + 10k + 1 = (9k+1)(k+1)$

$\Delta \geq 0$ for $k \leq -1$ or $k \geq -\frac{1}{9}$

aii

The curve of $y = \frac{x+2}{x^2-x-2}$ has tangent $y = k$ for $k = -1$ or $\frac{1}{9}$

$y(1) = -\frac{3}{2}$, indicating that the range lies outside the interval between these.

The range of $\mathrm{f}(x)$ is $\mathrm{f}(x) \geq -\frac{1}{9}$ or $\mathrm{f}(x) \leq -1$

b

Vertical asymptotes occur at the roots of the denominator

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

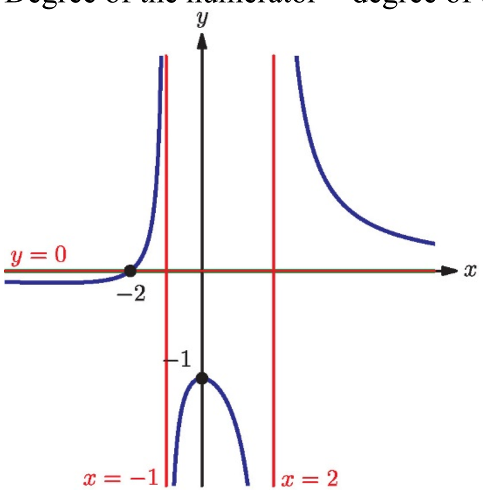
The asymptotes are $x = 2$ and $x = -1$

$\mathrm{f}(0) = -1$ so the y-intercept is at $(0, -1)$

The x-intercepts occur at the roots of the numerator: $x + 2 = 0$ so $(-2, 0)$ is the only one.

c

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



16

$$y = \frac{x - a}{(x - b)(x - c)}$$

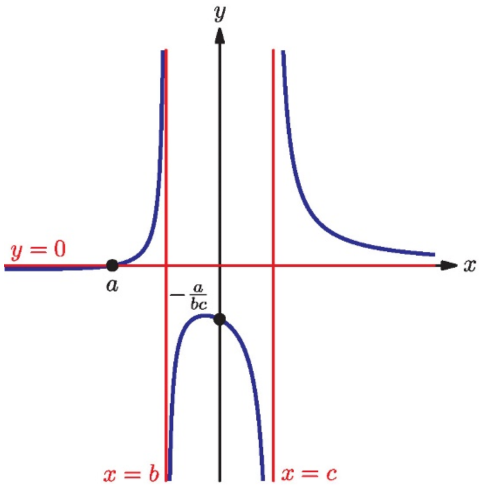
Vertical asymptotes $x = b, x = c$

x-intercept $(a, 0)$

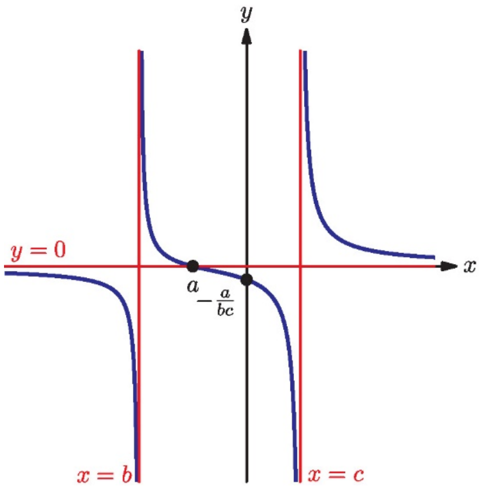
y-intercept $\left(0, -\frac{a}{bc}\right)$

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$

a



b



17 **a**

$$f(x) = \frac{x^2 - 6x + 10}{x - 3} = \frac{x(x - 3) - 3(x - 3) + 1}{x - 3} = x - 3 + \frac{1}{x - 3}$$

As $x \to \pm\infty, y \to x - 3$ so $y = x - 3$ is an oblique asymptote of the curve.

$A = 1, B = -3$

b

Solving intersections of $y = \mathrm{f}(x)$ and $y = k$:

$x^2 - 6x + 10 = k(x - 3)$

$x^2 - (6 + k)x + (10 + 3k) = 0$

This has real roots when discriminant $\Delta \geq 0$

$\Delta = (6 + k)^2 - 4(1)(10 + 3k)$

$\quad = k^2 - 4$

$\Delta = 0$ when $k = \pm 2$

The curve has turning points at $y = \pm 2$

$k = 2: x^2 - 8x + 16 = 0 \Rightarrow x = 4$

$k = -2: x^2 - 4x + 4 = 0 \Rightarrow x = 2$

Turning points are $(2, -2)$ and $(4, 2)$

c

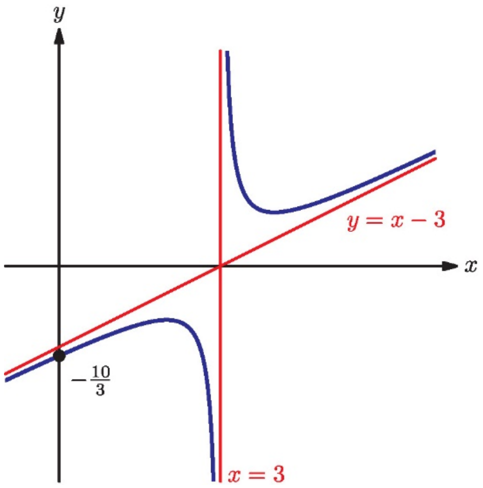
Vertical asymptote(s) at root(s) of denominator: $x - 3 = 0 \Rightarrow x = 3$

$\mathrm{f}(0) = -\frac{10}{3}$

x-intercept at roots of numerator: $(x - 3)^2 + 1 = 0$ has no real solution so there are no x-intercepts.

Axis intercept is $\left(0, -\frac{10}{3}\right)$

d



18 **a**

$$\frac{2x^2 - x - 3}{2x - 5} = \frac{(2x^2 - 5x) + 4x - 3}{2x - 5} = x + \frac{4x - 3}{2x - 5} = x + \frac{2(2x - 5) + 7}{2x - 5}$$

$$= x + 2 + \frac{7}{2x - 5}$$

$A = 1, B = 2, C = 7$

b

As $x \to \pm\infty, y \to x + 2$ so $y = x + 2$ is an oblique asymptote of the curve.

c

$f(x) = k$

$2x^2 - x - 3 = k(2x - 5)$

$2x^2 - (1 + 2k)x + (5k - 3) = 0$

This has real roots when discriminant $\Delta \geq 0$

$\Delta = (1 + 2k)^2 - 4(2)(5k - 3)$

$\quad = 4k^2 - 36k + 25$

$\Delta = 0$ when $k = \dfrac{36 \pm \sqrt{36^2 - 400}}{8} = \dfrac{9}{2} \pm \sqrt{14}$

The curve has turning points at $y = \dfrac{9}{2} \pm \sqrt{14}$

$f(0) = \dfrac{3}{5} < \dfrac{9}{2} - \sqrt{14}$ so the curve lies outside the interval $\dfrac{9}{2} - \sqrt{14} < y < \dfrac{9}{2} + \sqrt{14}$

The range is $f(x) \leq \dfrac{9}{2} - \sqrt{14}$ or $f(x) \geq \dfrac{9}{2} + \sqrt{14}$

d

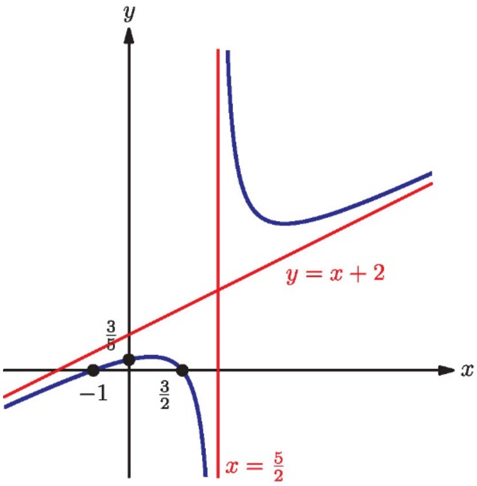
Vertical asymptote(s) at root(s) of denominator: $2x - 5 = 0 \Rightarrow x = \dfrac{5}{2}$

$f(0) = \dfrac{3}{5}$

x-intercept at roots of numerator: $(2x - 3)(x + 1) = 0$

Axis intercepts are $\left(\dfrac{3}{2}, 0\right), (-1, 0), \left(0, \dfrac{3}{5}\right)$

e



19

$f(x) = \dfrac{x + c}{x^2 - 3x - c}$ has range $f(x) \in \mathbb{R}$

$f(x) = k$ has a solution for all real values k

$x + c = k(x^2 - 3x - c)$

$kx^2 - (3k + 1)x - c(k + 1) = 0$

Require that c is such that x has at least one real solution for any value of k.

The quadratic has real roots when discriminant $\Delta \geq 0$

$\Delta = (3k + 1)^2 + 4(k)c(k + 1)$

$\quad = k^2(9 + 4c) + (6 + 4c)k + 1$

Require that $\Delta \geq 0$, for all k. Since this is a positive quadratic, this will be the case if there is no real root.

$$k = \frac{-(6+4c) \pm \sqrt{(6+4c)^2 - 36 - 16c}}{2(9+4c)} = \frac{-3-2c \pm \sqrt{4c^2+8c}}{9+4c}$$

For this not to have distinct real solutions, $4c^2 + 8c \leq 0$

$4c(c+2) \leq 0$

$-2 \leq c \leq 0$

Considering the boundary cases:

If $c = 0$ then $f(x) = \frac{x}{x^2 - 3x}$, which has range $f(x) \neq 0$, so $c = 0$ is not valid for the condition.

If $c = -2$ then $f(x) = \frac{x-2}{x^2 - 3x + 2}$

$f(x) = \frac{(x-2)}{(x-1)(x-2)}$ has range $f(x) \neq 0$, so $c = -2$ is also not valid for the condition.

Hence the solution is $-2 < c < 0$

20 **a**

$$f(x) = \frac{x^2 + 2ax + a^2 - 1}{x+a} = \frac{x(x+a) + a(x+a) - 1}{x+a} = x + a - \frac{1}{x+a}$$

As $x \to \pm\infty, y \to x + a$ so $y = x + a$ is an oblique asymptote of the curve.

b

Consider solutions to $f(x) = k$:

$x^2 + 2ax + a^2 - 1 = k(x+a)$

$x^2 + (2a - k)x + a^2 - ka - 1 = 0$

This has real roots when discriminant $\Delta \geq 0$

$\Delta = (2a-k)^2 - 4(1)(a^2 - ka - 1)$

$= k^2 + 4$

Since $\Delta > 0$ for all values of k, the range is $f(x) \in \mathbb{R}$

The function is the sum of a linear equation and a simple rational; if there is no interval not within the range then the function cannot have a turning point.

Knowing the shape the curve must take, the above argument is robust but may feel insufficient. Using calculus to show that there is no stationary point is also an option:

Quotient rule gives

$$f'(x) = \frac{(x+a)(2x+2a) - (x^2 + 2ax + a^2 - 1)}{(x+a)^2}$$
$$= \frac{x^2 + 2ax + a^2 + 1}{(x+a)^2}$$
$$= \frac{(x+a)^2 + 1}{(x+a)^2}$$

Stationary point occurs when $f'(x) = 0$

$(x+a)^2 = -1$ has no solution, so there is no stationary point.

Worked solutions

Alternatively, you could observe that the function is

$$f(x) = \frac{(x+a)^2 - 1}{x+a}$$

which is a translation a units to the left of the curve

$$g(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}$$

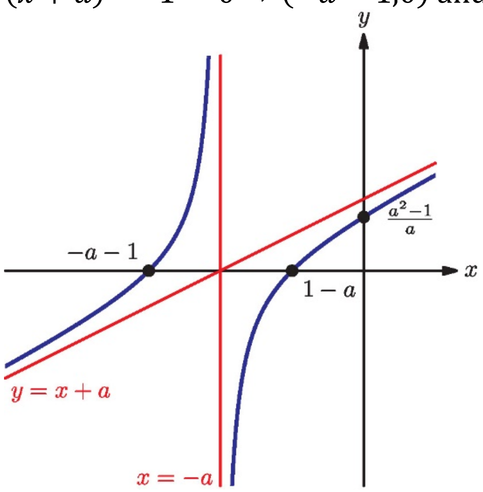
Then irrespective of the value of a, the curve will have the same characteristics as $g(x)$ which has no stationary point.

c

Vertical asymptote(s)at root(s)of denominator: $x + a = 0 \Rightarrow x = -a$

When $x = 0, y = \frac{a^2 - 1}{a}$

x-intercept at root of numerator:
$(x+a)^2 - 1 = 0 \Rightarrow (-a-1,0)$ and $(-a+1,0)$



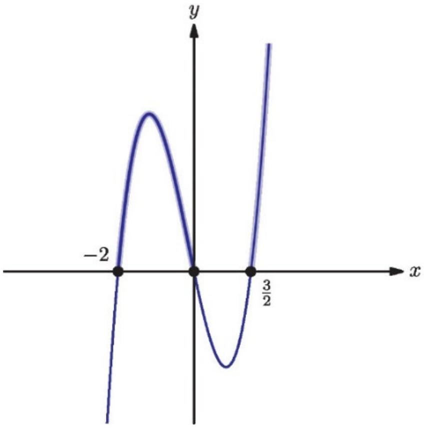
Exercise 7B

11

$2x^3 + x^2 - 6x > 0$

$x(2x - 3)(x + 2) > 0$

Positive cubic with three distinct roots; the value will be greater than zero between the first and second root and for values greater than the third root.



$-2 < x < 0$ or $x > \dfrac{3}{2}$

12 **a**

Let $f(x) = 2x^3 + x^2 - 7x - 6$

$f(2) = 2(8) + 4 - 7(2) - 6 = 0$ so by the factor theorem, $(x - 2)$ is a factor of $f(x)$

b

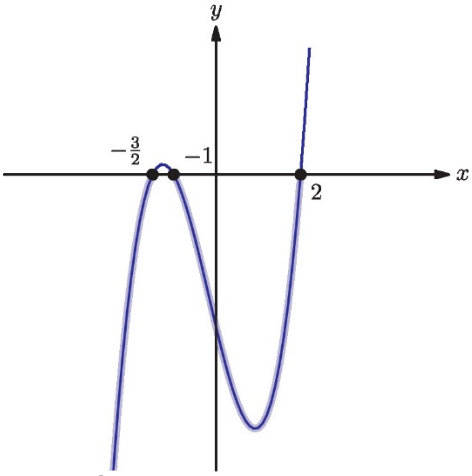
$3x^3 + 2x^2 \leq x^3 + x^2 + 7x + 6$

$2x^3 + x^2 - 7x - 6 \leq 0$

$(x - 2)(2x^2 + 5x + 3) \leq 0$

$(x - 2)(2x + 3)(x + 1) \leq 0$

Positive cubic with three distinct roots; the value will be less than zero for values less than the first root and between the second and third roots.



$x \leq -\dfrac{3}{2}$ or $-1 \leq x \leq 2$

13 **a**

Let $f(x) = 2x^3 + 11x^2 + 12x - 9$

$f(-3) = 2(-27) + 11(9) + 12(-3) - 9 = 0$ so by the factor theorem, $(x + 3)$ is a factor of $f(x)$

b

$11x^2 - 4 > 5 - 12x - 2x^3$

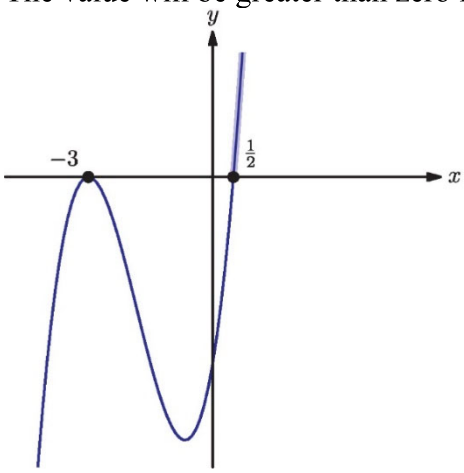
$2x^3 + 11x^2 + 12x - 9 > 0$

$(x + 3)(2x^2 + 5x - 3) > 0$

$(x + 3)(2x - 1)(x + 3) > 0$

Positive cubic with a repeated root at $x = -3$ and a single root at $x = 0.5$.

The value will be greater than zero for values greater than the third root.

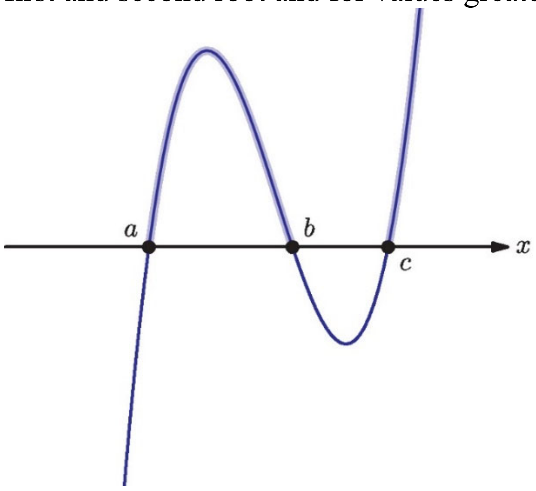


$x > \frac{1}{2}$

14

Let $f(x) = (x - a)(x - b)(x - c)$

Positive cubic with three distinct roots; the value will be greater than zero between the first and second root and for values greater than the third root.

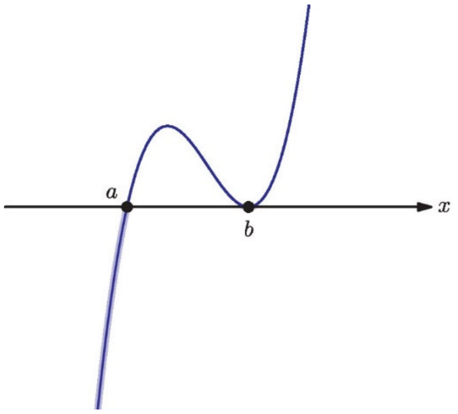


$a < x < b$ or $x > c$

15

Let $f(x) = (x - a)(x - b)^2$

Positive cubic with a single root at $x = a$ and a repeated root at $x = b > a$ the value will be less than zero for values less than the first root.



$x < a$

16

Let $f(x) = x^4 - 4x^2 + 3x + 1$

From the calculator, $f(x) \leq 0$ for $-2.26 \leq x \leq -0.251$

17

Let $f(x) = 2x^5 - 6x^4 + 8x^2 - 1$

From the calculator, $f(x) \geq 0$ for $-0.933 \leq x \leq -0.377$ or $0.371 \leq x \leq 1.76$ or $x > 2.18$

18

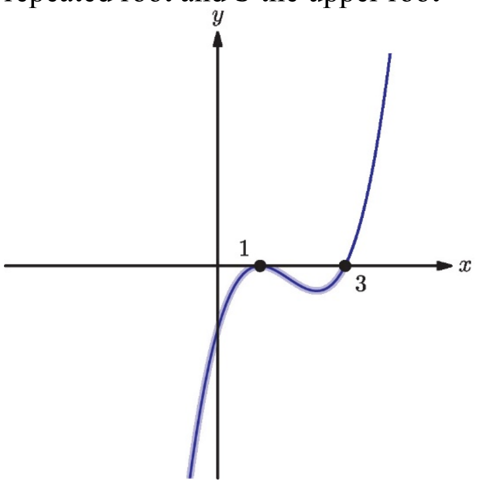
Let $f(x) = 3\ln(x^2 + 1) - x - 2$

From the calculator, $f(x) < 0$ for $-0.727 < x < 1.48$ or $x > 13.7$

19

Let $f(x) = x^3 + bx^2 + cx + d - 2$

$f(x)$ is a positive cubic. If the solutions to $f(x) < 0$ are $x < 3, x \neq 1$ then 1 must be a repeated root and 3 the upper root



$$\begin{aligned} f(x) &= (x-1)^2(x-3) \\ &= (x^2 - 2x + 1)(x - 3) \\ &= x^3 - 5x^2 + 7x - 3 \end{aligned}$$

$b = -5, c = 7, d = -1$

20

Let $\mathrm{f}(x) = ax^3 + bx^2 + cx + d - 3$

$\mathrm{f}(x) = 0$ at $-4, -1, 1.5$

$\mathrm{f}(x)$ is a cubic which is positive for x less than the least root or between the two greater roots so $\mathrm{f}(x)$ must be a negative cubic.

$\mathrm{f}(x) = a(x+4)(x+1)\left(x - \frac{3}{2}\right)$

Require that all coefficients are integers, so a must be an even value (and negative, by the above argument).

If $a = -2k$ then

$$\begin{aligned}\mathrm{f}(x) &= -2k(x+4)(x+1)\left(x-\frac{3}{2}\right)\\ &= -k(x^2+5x+4)(2x-3)\\ &= k(-2x^3-7x^2+7x+12)\end{aligned}$$

$a = -2k, b = -7k, c = 7k, d = 12k + 3$ for any positive integer k.

If $|a|$ is the least possible then $k = 1$

$a = -2, b = -7, c = 7, d = 15$

21

Let $\mathrm{f}(x) = \frac{3x}{(x+3)(x-2)} - 2^x$

Vertical asymptotes are $x = -3$ and $x = 2$

From calculator, $\mathrm{f}(x) \geq 0$ for $-3 < x \leq -1$ or $2 < x \leq 2.27$

22

Let $\mathrm{f}(x) = \frac{x-2}{(3x-4)(x+2)} - \ln(x+4)$

Vertical asymptotes are $x = -4, x = -2$ and $x = \frac{4}{3}$

The graph is not defined for $x \leq -4$

From calculator, $\mathrm{f}(x) \leq 0$ for $-3.26 \leq x < -2$ or $-1.54 \leq x \leq 1.29$ or $x > \frac{4}{3}$

23

A function is strictly decreasing when its derivative has a negative value.

$\mathrm{f}'(x) = 24x^2 - 4x^3 = 4x^2(6 - x)$

$\mathrm{f}'(x) < 0$ for $x > 6$

Properly, the question asked for ‘decreasing’ not ‘strictly decreasing’. This would also allow for $\mathrm{f}'(x) = 0$ to be within the solution, so the answer would be $x = 0$ or $x \geq 6$.

This nuance is not detailed within the IB syllabus.

24

A function is strictly increasing when its derivative has a positive value.

$$\begin{aligned}\mathrm{f}'(x) &= 4x^3 - 12x^2 - 4x + 12\\ &= 4(x^3 - 3x^2 - x + 3)\\ &= 4(x^2-1)(x-3)\\ &= 4(x-1)(x+1)(x-3)\end{aligned}$$

$\mathrm{f}'(x)$ is a positive cubic with roots at ± 1 and 3

$\mathrm{f}'(x) > 0$ for $-1 < x < 1$ or $x > 3$

As with question 23 above, since the question says ‘increasing’ not ‘strictly increasing’ the correct answer would actually be $-1 \leq x \leq 1$ or $x \geq 3$

25 **a**

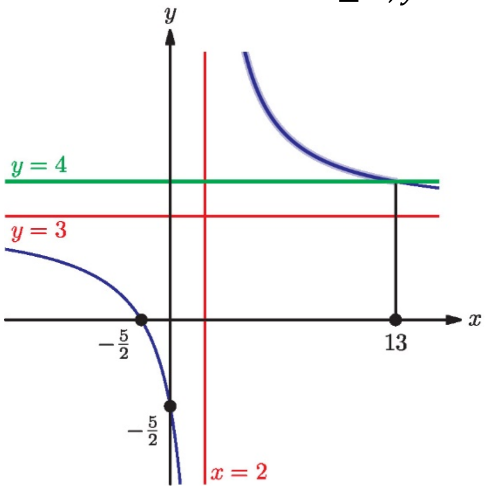
$$y = \frac{3x+5}{x-2}$$

Vertical asymptote at root(s) of denominator: $x = 2$

x-intercept(s) at root(s) of numerator: $x = -\frac{5}{3}$ so intercept is $\left(-\frac{5}{3}, 0\right)$

y-intercept when $x = 0$: $\left(0, -\frac{5}{2}\right)$

Degree of numerator equals degree of denominator. Horizontal asymptote at ratio of lead coefficients. As $x \to \pm\infty, y \to 3$



b

Single intersection: $\frac{3x+5}{x-2} = 4 \Rightarrow 3x+5 = 4x-8 \Rightarrow x = 13$

Solution is $2 < x \leq 13$

26 **a**

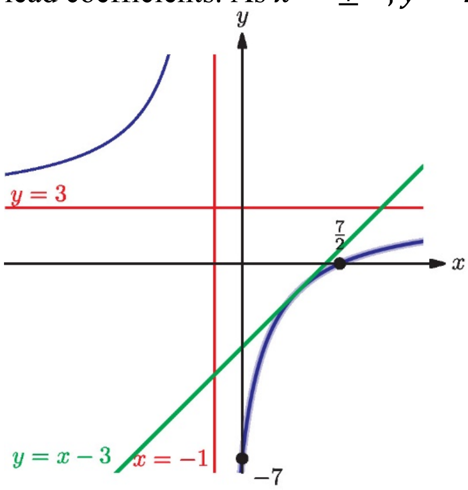
$$y = \frac{2x-7}{x+1}$$

Vertical asymptote at root(s) of denominator: $x = -1$

x-intercept(s) at root(s) of numerator: $x = \frac{7}{2}$ so intercept is $\left(\frac{7}{2}, 0\right)$

y-intercept when $x = 0$: $(0, -7)$

Degree of numerator equals degree of denominator. Horizontal asymptote at ratio of lead coefficients. As $x \to \pm\infty, y \to 2$



b

Checking for an intersection point: $\dfrac{2x-7}{x+1} = x-3$

$x^2 - 2x - 3 = 2x - 7$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

The line is tangent to the curve.

$\dfrac{2x-1}{x+5} < x - 3$ has solution $-1 < x < 2$ or $x > 2$

27 **a**

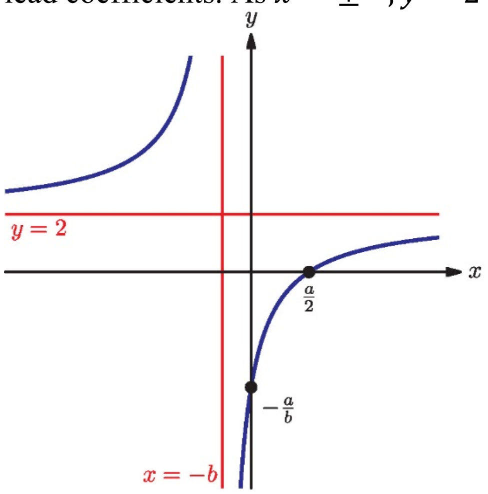
$y = \dfrac{2x-a}{x+b}$

Vertical asymptote at root(s) of denominator: $x = -b$

x-intercept(s) at root(s) of numerator: $x = \frac{a}{2}$ so intercept is $\left(\frac{a}{2}, 0\right)$

y-intercept when $x = 0$: $\left(0, -\frac{a}{b}\right)$

Degree of numerator equals degree of denominator. Horizontal asymptote at ratio of lead coefficients. As $x \to \pm\infty, y \to 2$



b

$\dfrac{2x-a}{x+b} = 3$ has a single solution:

$3x + 3b = 2x - a$

$x = -a - 3b$

$\dfrac{2x-a}{x+b} > 3$ has solution $-a - 3b < x < -b$

28

Finding boundaries to the solution set:

The vertical asymptote of the rational function is $x = -\frac{1}{p}$

$$\frac{16x+1}{px+1} = x+4$$

$$16x + 1 = px^2 + (4p+1)x + 4$$

$$px^2 + (4p-15)x + 3 = 0$$

The expression on the left is a quadratic with roots $x = \alpha$ or β (with $\alpha \le \beta$)

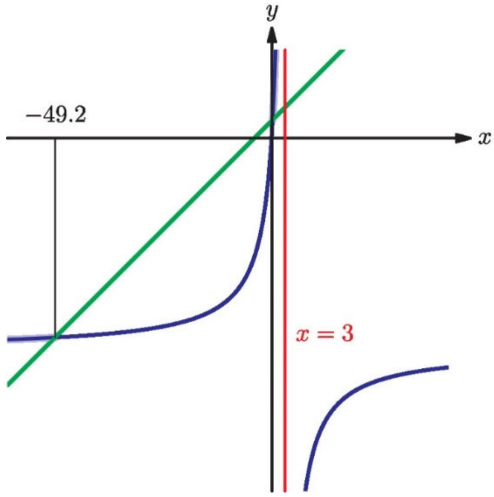
The solution to the inequality therefore has boundaries $x = -\frac{1}{p}, x = \alpha$ and $x = \beta$

The solution has the form $x < q$ or $r < x < 3$

Then $-\frac{1}{p} = 3$ or $\beta = 3$

$-\frac{1}{p} = 3 : p = -\frac{1}{3}$, the quadratic is $-\frac{1}{3}x^2 - \frac{49}{3}x + 3 = 0$

$x^2 + 49x - 9 = 0$ has roots -49.2 and 0.183



The solution to $\frac{16x+1}{px+1} > x + 4$ would be

$x < -49.2$ or $0.183 < x < 3$

$p = -\frac{1}{3}, q = -49.2, r = 0.183$

If 3 is a root of the quadratic then $(x-3)$ is a factor

so $px^2 + (4p-15)x + 3 = (x-3)(ax+b)$

Comparing coefficients:

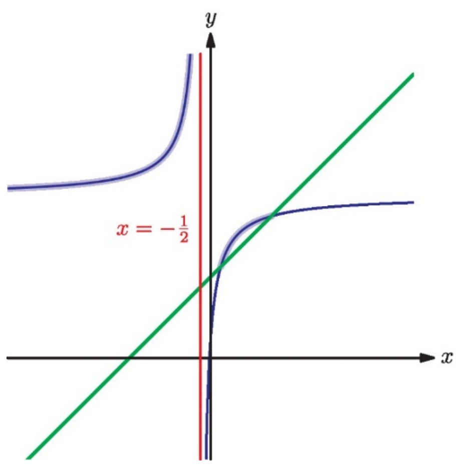
$x^0: b = -1$

$x^1: 4p - 15 = b - 3a = -1 - 3a$

$x^2: p = a$

Substituting: $4a - 15 = -1 - 3a \Rightarrow a = 2$

and the other factor is $(2x - 1)$



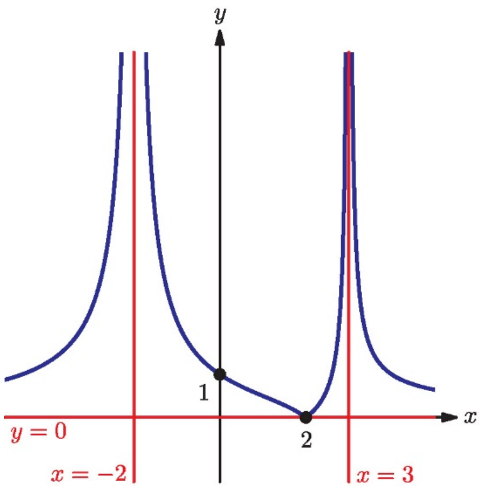
The solution to $\frac{16x+1}{px+1} > x + 4$ would be

$x < -0.5$ or $0.5 < x < 3$

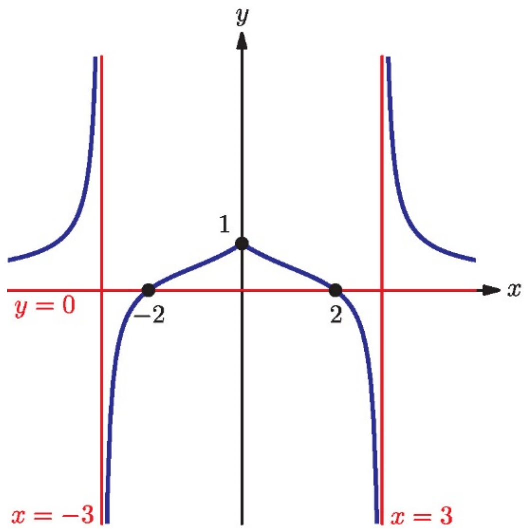
$p = 2, q = -0.5, r = 0.5$

Exercise 7C

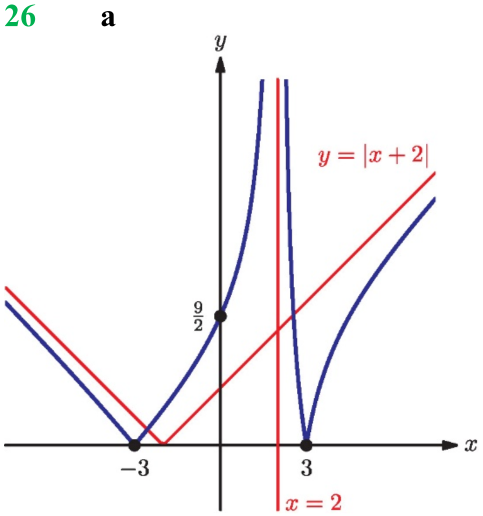
25 **a**



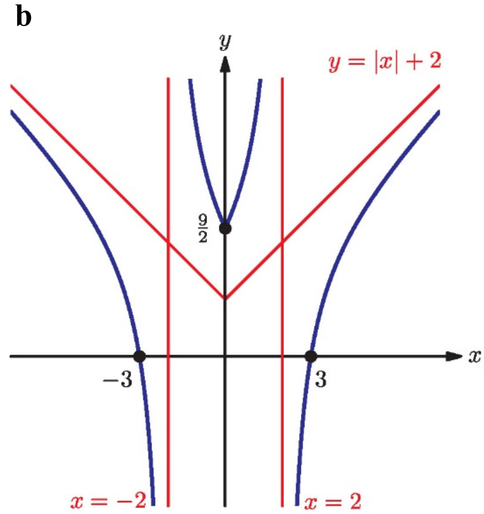
b



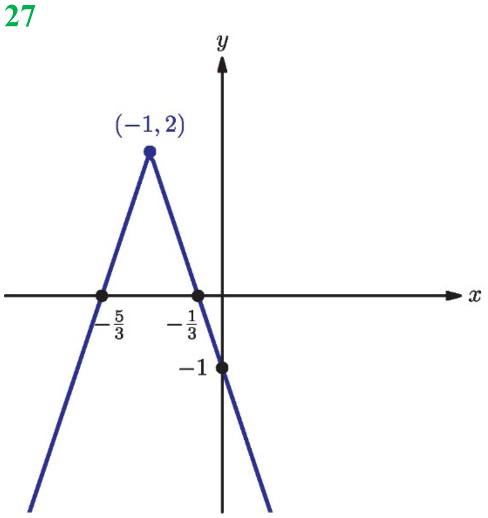
26 **a**



b



27



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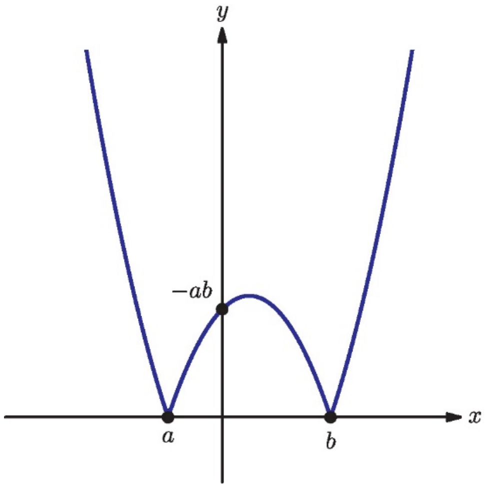
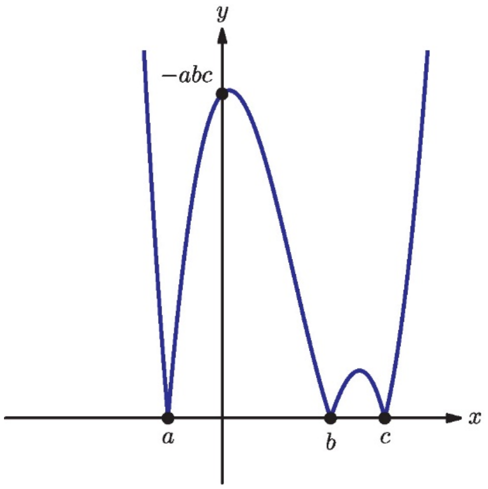


29

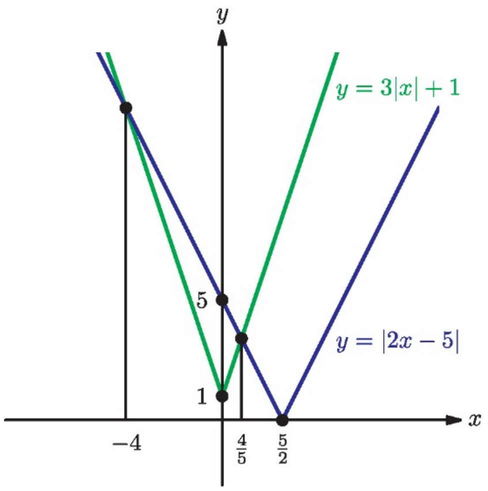


30 From calculator: $-1.28 \leq x \leq 0.720$

31 From calculator: $x < -4.80$ or $-3.32 < x < 4.80$

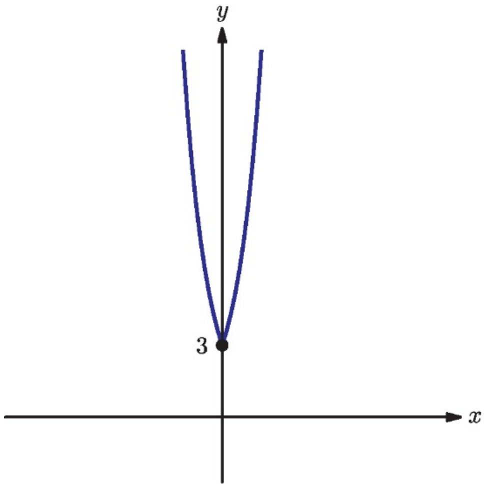
32 From calculator: $x < -0.146$ or $0.180 < x < 0.967$

33 **a**



b $x < -4$ or $x > \frac{4}{5}$

34 **a**



b

$3e^{|x|} > 5$

$|x| > \ln\left(\frac{5}{3}\right)$

$x < \ln\left(\frac{3}{5}\right)$ or $x > \ln\left(\frac{5}{3}\right)$

35

As $x \to \pm\infty, y \to -\infty \Rightarrow a < 0$

Point of symmetry is at $x = -3$ so $b = 3$

$$y(0) = -1 = a|b| + c \quad (1)$$

$$y\left(-\frac{11}{2}\right) = 0 = c + a\left|b - \frac{11}{2}\right| \quad (2)$$

$$y\left(-\frac{1}{2}\right) = 0 = c + a\left|b - \frac{1}{2}\right| \quad (3)$$

(1): $3a + c = -1 \quad (4)$

(2): $2.5a + c = 0 \quad (5)$

$(4) - (5)$: $0.5a = -1 \Rightarrow a = -2$

$a = -2, b = 3, c = 5$

36

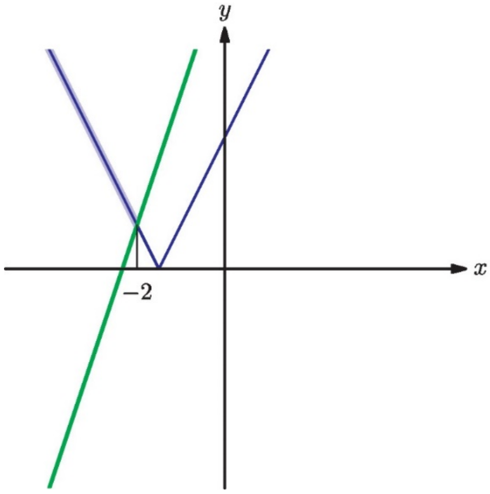
$|2x+3| = 3x+7$

Left arm of the modulus graph: $2x+3<0$

$2x+3 = -3x-7 \Rightarrow 5x = -10 \Rightarrow x = -2$

Right arm of the modulus graph: $2x+3>0$

$2x+3 = 3x+7 \Rightarrow x = -4$ contradicts condition; no intersection



$|2x+3| > 3x+7$

$x < -2$

37

$|x^2-3x-5| = 3-x$

Left arm of the modulus graph: $x^2-3x-5<0$

$x^2-3x-5 = x-3 \Rightarrow x^2-4x-2 = 0 \Rightarrow x = 2 \pm \sqrt{6}$

Only $x = 2-\sqrt{6}$ satisfies the condition.

Right arm of the modulus graph: $x^2-3x-5>0$

$x^2-3x-5 = 3-x \Rightarrow x^2-2x-8 = 0 \Rightarrow (x-4)(x+2) = 0 \Rightarrow x = 4 \text{ or} -2$

Only $x = -2$ satisfies the condition.

The solutions are $x = -2$ or $2-\sqrt{6}$

38

Boundary points when $|x^2-5x+4| = 2$

Left arm of the modulus graph: $x^2-5x+4<0$

$x^2-5x+4 = -2 \Rightarrow x^2-5x+6 = 0 \Rightarrow x = 2 \text{ or } 3$

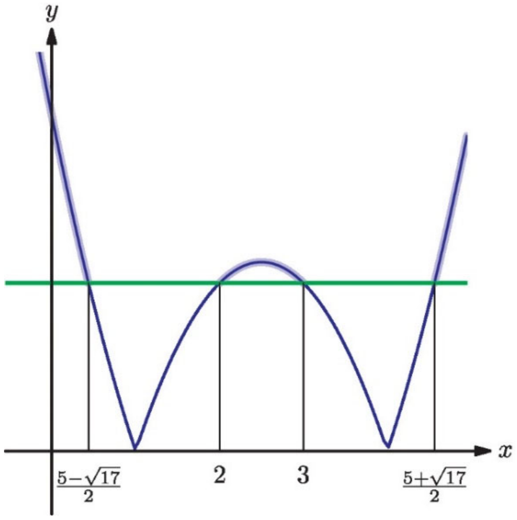
Both values satisfy the condition

Right arm of the modulus graph: $x^2-5x+4>0$

$$x^2-5x+4 = 2 \Rightarrow x^2-5x+2 = 0 \Rightarrow x = \frac{5\pm\sqrt{17}}{2}$$

Both values satisfy the condition

The solutions are $x < \frac{5-\sqrt{17}}{2}$ or $2 < x < 3$ or $x > \frac{5+\sqrt{17}}{2}$



$|x^2 - 5x + 4| > 2$

39 Split into three regions:

Case 1: $x < -1$

$|x + 1| + |x - 1| = -1 - x + 1 - x = -2x$

$-2x = x + 4$

$x = -\frac{4}{3}$

Case 2: $-1 < x < 1$

$|x + 1| + |x - 1| = x + 1 + 1 - x = 2$

$2 = x + 4$

$x = -2$ which contradicts the case presumption

Case 3: $x > 1$

$|x + 1| + |x - 1| = x + 1 + x - 1 = 2x$

$2x = x + 4$

$x = 4$

The solutions are $x = -\frac{4}{3}$ or 4

40 Split into three regions:

Case 1: $x < \frac{1}{3}$

$|3x - 1| = 1 - 3x;\ x + |2 - x| = x + 2 - x = 2$

$1 - 3x = 2$

$x = -\frac{1}{3}$

Case 2: $\frac{1}{3} < x < 2$

$|3x - 1| = 3x - 1;\ x + |2 - x| = x + 2 - x = 2$

$3x - 1 = 2$

$x = 1$

Case 3: $x > 2$

$|3x - 1| = 3x - 1;\ x + |2 - x| = x + x - 2 = 2x - 2$

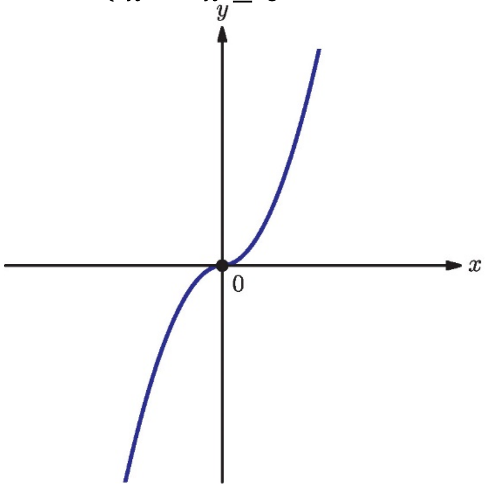
$3x - 1 = 2x - 2$

$x = -1$ which contradicts the case presumption

The solutions are $x = -\frac{1}{3}$ or 1

41 **a**

$$x|x| = \begin{cases} -x^2 & x \leq 0 \\ x^2 & x \geq 0 \end{cases}$$



b

$x|x| = kx$

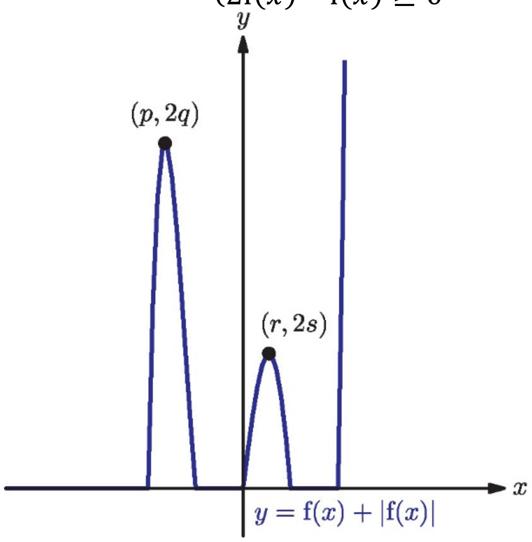
$x < 0: -x^2 = kx \Rightarrow x = 0, -k$

$x > 0: x^2 = kx \Rightarrow x = 0, k$

Solutions: $x = 0, \pm k$

42

$$\mathrm{f}(x) + |\mathrm{f}(x)| = \begin{cases} 0 & \mathrm{f}(x) \leq 0 \\ 2\mathrm{f}(x) & \mathrm{f}(x) \geq 0 \end{cases}$$



43 Split into cases:

<u>Case 1: $x < -a^2$</u>

$|x + a^2| = -a^2 - x; |x - 2a^2| = 2a^2 - x$

$-a^2 - x = 2a^2 - x$

No solution

<u>Case 2: $-a^2 < x < 2a^2$</u>

$|x + a^2| = x + a^2; |x - 2a^2| = 2a^2 - x$

$x + a^2 = 2a^2 - x$

$2x = a^2$

$x = \frac{a^2}{2}$

<u>Case 3: $x > 2a^2$</u>

$|x + a^2| = x + a^2; |x - 2a^2| = x - 2a^2$

$x + a^2 = x - 2a^2$

No solution

44

Let $\mathrm{f}(x) = x^2 + 4x - 7 = (x + 2)^2 - 11$

$\mathrm{f}(x)$ has range $\mathrm{f}(x) > -11$

For $|\mathrm{f}(x)| = k$ to have four solutions, require that $\mathrm{f}(x) = k$ has two distinct solutions and also $\mathrm{f}(x) = -k$ has two distinct solutions; this can only happen for $0 < k < 11$

45

Let $\mathrm{f}(x) = x^3 - 12x + 4$

$\mathrm{f}'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$

$\mathrm{f}(x)$ is a positive cubic, so the stationary point at $(-2, 20)$ is a local maximum and $(2, -12)$ is a local minimum.

$\mathrm{f}(x) = k$ has

one solution for $k < -12$ or $k > 20$

two solutions for $k = -12$ or 20

three solutions for $-12 < k < 20$

For $|\mathrm{f}(x)| = k$ to have four solutions, require that

$\mathrm{f}(x) = k$ has one solution and $\mathrm{f}(x) = -k$ has three distinct solutions: No such values

or

$\mathrm{f}(x) = k$ has two distinct solutions and $\mathrm{f}(x) = -k$ has two distinct solutions: No such values

or

$\mathrm{f}(x) = k$ has three distinct solutions and $\mathrm{f}(x) = -k$ has one solution: $12 < k < 20$

There are four solutions for $12 < k < 20$

Exercise 7D

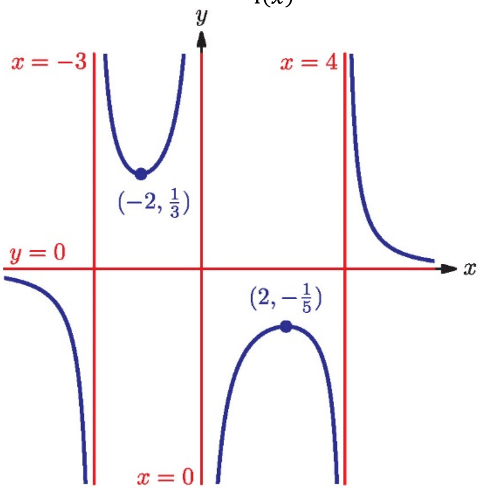
17 **a**

$\frac{1}{\mathrm{f}(x)}$ has vertical asymptotes at roots of $\mathrm{f}(x)$: $x = -3, x = 0, x = 4$

Local minima of $\mathrm{f}(x)$ are maxima of $\frac{1}{\mathrm{f}(x)}$ and vice versa:

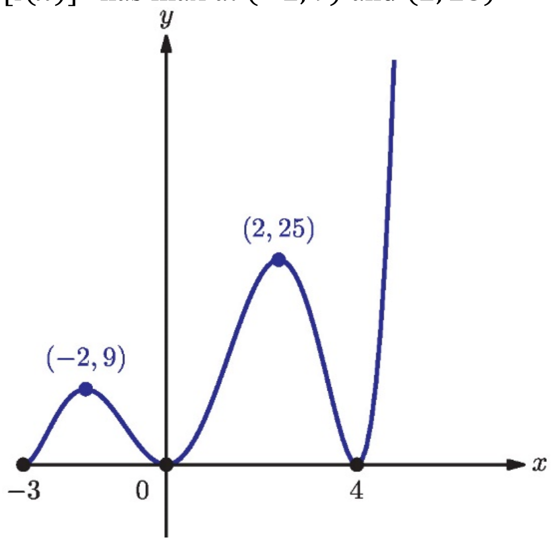
$\frac{1}{\mathrm{f}(x)}$ has min at $\left(-2, \frac{1}{3}\right)$,max at$\left(2, -\frac{1}{5}\right)$

If $\mathrm{f}(x) \to \pm\infty$ then $\frac{1}{\mathrm{f}(x)} \to 0$



b

$[\mathrm{f}(x)]^2$ has the same roots as $\mathrm{f}(x)$, intersections with the axis become minima.
Local minima at $(-3, 0), (0, 0), (4, 0)$
Positive stationary points keep the same character, negative stationary points take the opposite character:
$[\mathrm{f}(x)]^2$ has max at $(-2, 9)$ and $(2, 25)$



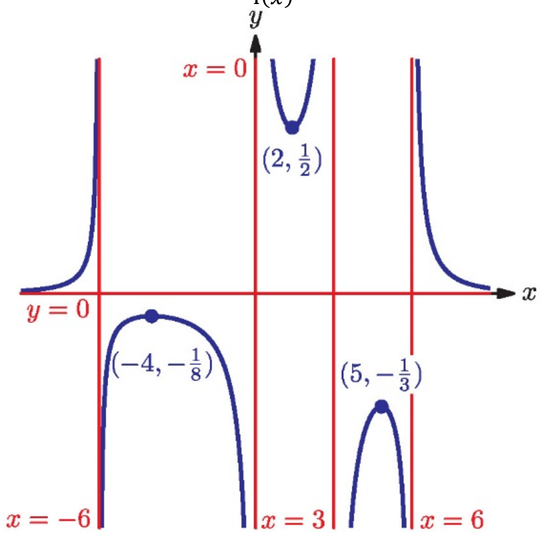
18 **a**

$\frac{1}{f(x)}$ has vertical asymptotes at roots of $f(x)$: $x = -6, x = 0, x = 3, x = 6$

Local minima of $f(x)$ are maxima of $\frac{1}{f(x)}$ and vice versa:

$\frac{1}{f(x)}$ has max at $\left(-4, -\frac{1}{8}\right)$ and $\left(5, -\frac{1}{3}\right)$, min at $\left(2, \frac{1}{2}\right)$

If $f(x) \to \pm\infty$ then $\frac{1}{f(x)} \to 0$



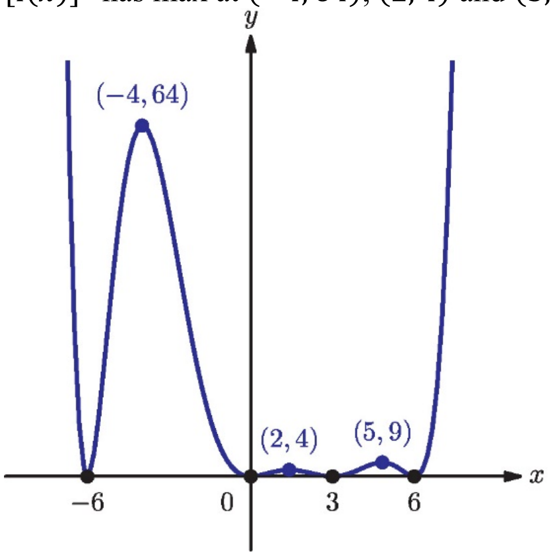
b

$[f(x)]^2$ has the same roots as $f(x)$, intersections with the axis become minima.

Local minima at $(-6, 0), (0, 0), (3, 0), (6, 0)$

Positive stationary points keep the same character, negative stationary points take the opposite character:

$[f(x)]^2$ has max at $(-4, 64)$, $(2, 4)$ and $(5, 9)$



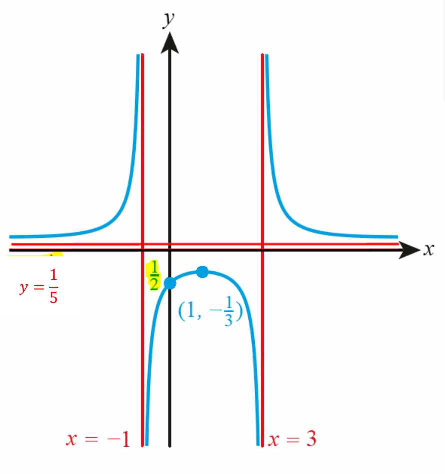
19 **a**

$\frac{1}{f(x)}$ has vertical asymptotes at roots of $\mathrm{f}(x)$: $x = -1, x = 3$

Local minima of $f(x)$ are maxima of $\frac{1}{\mathrm{f}(x)}$ and vice versa: $\frac{1}{f(x)}$ has max at $\left(1, -\frac{1}{3}\right)$

$\mathrm{f}(0) = -2$ so $\frac{1}{\mathrm{f}(0)} = -\frac{1}{2}$

$f(x) \to 5$ so $\frac{1}{\mathrm{f}(x)} \to \frac{1}{5}$



b

$[\mathrm{f}(x)]^2$ has the same roots as $\mathrm{f}(x)$, intersections with the axis become minima.

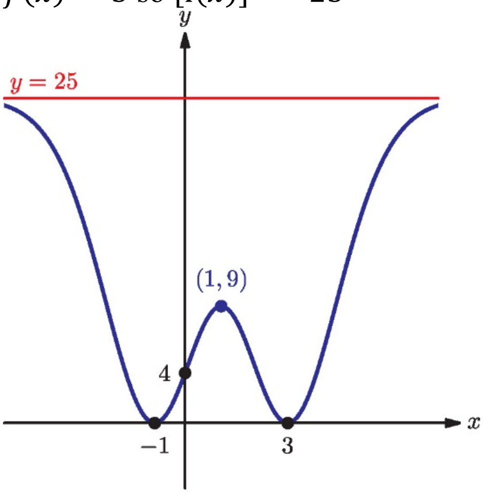
Local minima at $(-1, 0), (3, 0)$

Positive stationary points keep the same character, negative stationary points take the opposite character:

$\frac{1}{f(x)}$ has max at $(1, 9)$

$\mathrm{f}(0) = -2$ so $[\mathrm{f}(x)]^2 = 4$

$f(x) \to 5$ so $[\mathrm{f}(x)]^2 \to 25$



20 **a** $\left(3, -\frac{1}{4}\right)$

b $(3, 16)$

c $\frac{1}{2}x + 2 = 3 \Rightarrow x = 2$

$(2, -4)$

21

x replaced by $x-\frac{\pi}{4}$: translation $\frac{\pi}{4}$ to the right

x replaced by $3x$: horizontal stretch with scale factor $\frac{1}{3}$

The total transformation is a translation $\begin{pmatrix}\frac{\pi}{4}\\0\end{pmatrix}$ followed by a horizontal stretch with scale factor $\frac{1}{3}$

22

x replaced by $x+3$: translation 3 to the left

x replaced by $\frac{2}{5}x$: horizontal stretch with scale factor $\frac{5}{2}$

The total transformation is a translation $\begin{pmatrix}-3\\0\end{pmatrix}$ followed by a horizontal stretch with scale factor $\frac{5}{2}$

23

$f_1(x)=3x^2+4x$

Horizontal stretch scale factor $\frac{1}{2}$: replace x with $2x$

$f_2(x)=12x^2+8x$

Translation $\begin{pmatrix}1\\0\end{pmatrix}$: Replace x with $(x-1)$

$f_3(x)=12(x-1)^2+8(x-1)=12x^2-16x+4$

24

$f_1(x)=f(x)$

Translation $\begin{pmatrix}4\\0\end{pmatrix}$: Replace x with $(x-4)$; $f_2(x)=f_1(x-4)$

$f_2(x)=f(x-4)$

Horizontal stretch scale factor $\frac{1}{2}$: replace x with $2x$; $f_3(x)=f_2(2x)$

$f_3(x)=f(2x-4)$

Translation $\begin{pmatrix}0\\3\end{pmatrix}$: $f_4(x)=f_3(x)+3$

$f_4(x)=f(2x-4)+3$

Vertical stretch scale factor 2: $f_5(x)=2f_4(x)$

$f_5(x)=2f(2x-4)+6$

25

$f_1(x)=f(x)$

Vertical stretch scale factor $\frac{1}{3}$: $f_2(x)=\frac{1}{3}f_1(x)$

$f_2(x)=\frac{1}{3}f(x)$

Translation $\begin{pmatrix}0\\-4\end{pmatrix}$: $f_3(x)=f_2(x)-4$

$f_3(x)=\frac{1}{3}f(x)-4$

Horizontal stretch scale factor 2: $f_4(x)=f_3\left(\frac{x}{2}\right)$

$f_4(x)=\frac{1}{3}f\left(\frac{x}{2}\right)-4$

Translation $\begin{pmatrix}-1\\0\end{pmatrix}$: Replace x with $(x+1)$; $f_5(x)=f_4(x+1)$

$f_5(x)=\frac{1}{3}f\left(\frac{x+1}{2}\right)-4$

26

A: Reflection in the y-axis. Replace x with $(-x)$; $g(x) = f(-x)$

B: Translation $\begin{pmatrix}5\\0\end{pmatrix}$. Replace x with $(x-5)$; $g(x) = f(x-5)$

a)

A then B: $g(x) = f\big(-(x-5)\big) = f(5-x)$

b)

B then A: $g(x) = f\big((-x)-5\big) = f(-x-5)$

27

$f(x)$ has vertical asymptotes at roots of $\frac{1}{f(x)}$: $x = \pm 4$

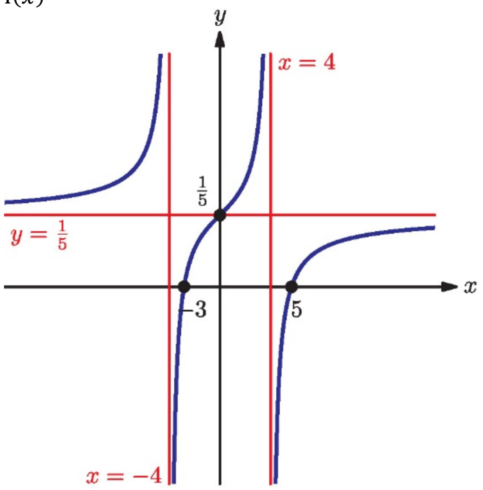
$f(x)$ has x-intercepts at vertical asymptotes of $\frac{1}{f(x)}$: $(-3, 0), (5, 0)$

Local minima of $\frac{1}{f(x)}$ are maxima of $f(x)$ and vice versa:

No local minima or maxima

If $\frac{1}{f(x)} \to 5$ as $x \to \pm\infty$ then $f(x) \to \frac{1}{5}$

$\frac{1}{f(x)}$ has y-intercept at $(5, 0)$ so $f(x)$ has y-intercept $\left(0, \frac{1}{5}\right)$



28

$[f(x)]^2$ has the same roots and vertical asymptotes as $f(x)$; intersections with the axis become minima.

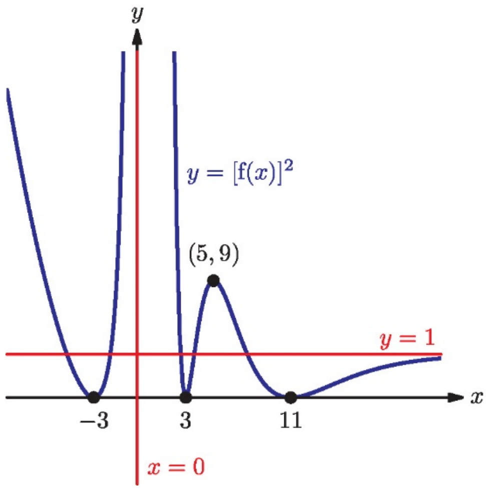
Local minima at $(\pm 3, 0), (11, 0)$, vertical asymptote $x = 0$.

Positive stationary points keep the same character, negative stationary points take the opposite character:

$[f(x)]^2$ has max at $(5, 9)$

$f(x) \to -1$ as $x \to \infty$ so $[f(x)]^2 \to 1$

$f(x) \to -\infty$ as $x \to -\infty$ so $[f(x)]^2 \to \infty$



29

Transforming $\mathrm{f}(x)$ to $\mathrm{g}(x) = \frac{1}{\mathrm{f}(ax-b)}$

$\mathrm{f}_1(x) = \mathrm{f}(x)$

$\mathrm{f}_2(x) = \mathrm{f}_1(x-b) = \mathrm{f}(x-b)$: Translation $\begin{pmatrix} b \\ 0 \end{pmatrix}$

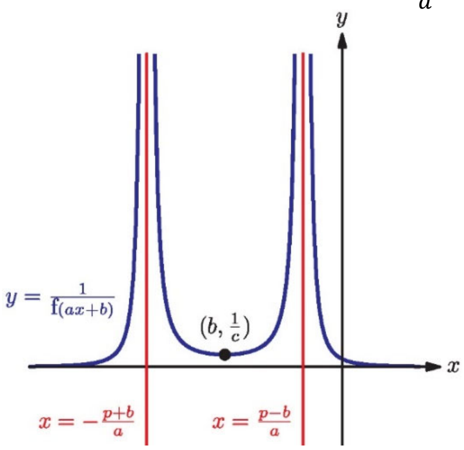
Roots are at $(b \pm p, 0)$, local maximum is at (b, c). Shape is the same.

$\mathrm{f}_3(x) = \mathrm{f}_2(ax) = \mathrm{f}(ax-b)$: Horizontal stretch with scale factor $\frac{1}{a}$

Roots are at $\left(\frac{b\pm p}{a}, 0\right)$, local maximum is at $\left(\frac{b}{a}, c\right)$. Shape is the same.

$\mathrm{f}_4(x) = \frac{1}{\mathrm{f}_3(x)} = \frac{1}{\mathrm{f}(ax-b)}$: Reciprocal transformation

Vertical asymptotes are at $x = \frac{b\pm p}{a}$, local minimum is at $\left(\frac{b}{a}, \frac{1}{c}\right)$. As $x \to \pm\infty, y \to 0$



30

Transforming $\mathrm{f}(x)$ to $\mathrm{g}(x) = [f(ax+b)]^2$

$\mathrm{f}_1(x) = \mathrm{f}(x)$

$\mathrm{f}_2(x) = \mathrm{f}_1(x+b) = \mathrm{f}(x+b)$: Translation $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

Intercept is at $(a-b, 0)$, vertical asymptote is $x = -b$. Shape is the same.

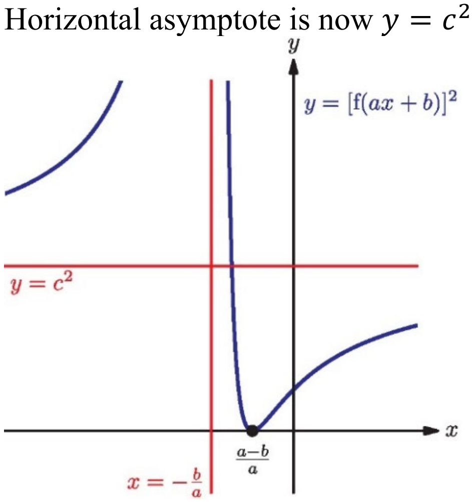
$\mathrm{f}_3(x) = \mathrm{f}_2(ax) = \mathrm{f}(ax+b)$: Horizontal stretch with scale factor $\frac{1}{a}$

Intercept is at $\left(\frac{a-b}{a}, 0\right)$, vertical asymptote is $x = -\frac{b}{a}$. Shape is the same.

$\mathrm{f}_4(x) = [\mathrm{f}_3(x)]^2 = [\mathrm{f}(ax+b)]^2$: Squaring the function

Intercept is at $\left(\frac{a-b}{a}, 0\right)$ is now a local minimum, vertical asymptote is $x = -\frac{b}{a}$.

Horizontal asymptote is now $y = c^2$



31

There are infinitely many possible answers to this question, due to the periodic nature of the sine curve. It would be expected that a student would either give a general answer or a sensible low absolute value answer.

The solution below approaches the problem as a graph transformation question; there are more direct approaches available and a student familiar with the sine curve might be expected to be able to read the values of a, b, c directly from the graph without this working, citing amplitude a, period $\frac{2\pi}{b}$ and left shift c.

Transforming $\mathrm{f}(x) = \sin x$ to $g(x) = a\sin(bx + c)$

$\mathrm{f}_1(x) = \sin(x)$

$\mathrm{f}_2(x) = \mathrm{f}_1(x + c) = \sin(x + c)$: Translation $\begin{pmatrix} -c \\ 0 \end{pmatrix}$

$\mathrm{f}_3(x) = \mathrm{f}_2(bx) = \sin(bx + c)$: Horizontal stretch with scale factor $\frac{1}{b}$

$\mathrm{f}_4(x) = a\mathrm{f}_3(x) = a\sin(bx + c)$: Vertical stretch with scale factor a

$\mathrm{g}(x)$ has period $\frac{5\pi}{3} - \left(-\frac{7\pi}{3}\right) = 4\pi$, $\sin x$ has period 2π

Horizontal stretch scale factor is $2 = \frac{1}{b} \Rightarrow b = \frac{1}{2}$

$\mathrm{g}(x)$ has amplitude $4 = a$

$\mathrm{g}\left(-\frac{\pi}{3}\right) = -4 = 4\sin\left(c - \frac{\pi}{6}\right)$

$\sin\left(c - \frac{\pi}{6}\right) = -1 = \sin\left(2n\pi - \frac{\pi}{2}\right)$ for integer n

$c = 2n\pi - \frac{\pi}{3}$

$a = 4, b = \frac{1}{2}, c = \left(2n - \frac{1}{3}\right)\pi$

An alternative solution set is $a = -4, b = \frac{1}{2}, c = 2n\pi + \frac{2\pi}{3}$

32

Transforming $\mathrm{f}(x) = \sin x$ to $g(x) = a + \sin(bx + c)$

$\mathrm{f}_1(x) = \sin(x)$

$\mathrm{f}_2(x) = \mathrm{f}_1(x + c) = \sin(x + c)$: Translation $\begin{pmatrix} -c \\ 0 \end{pmatrix}$

$\mathrm{f}_3(x) = \mathrm{f}_2(bx) = \sin(bx + c)$: Horizontal stretch with scale factor $\frac{1}{b}$

$\mathrm{f}_4(x) = a + \mathrm{f}_3(x) = a + \sin(bx + c)$: Translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$

$\mathrm{g}(x)$ has period $\frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$, $\sin x$ has period 2π

Horizontal stretch scale factor is $\frac{1}{3} = \frac{1}{b} \Rightarrow b = 3$

$\mathrm{g}(x)$ has central value $-1 = a$

$\mathrm{g}\left(-\frac{\pi}{3}\right) = -2 = -1 + \sin(c - \pi)$

$\sin(c - \pi) = -1 = \sin\left(2n\pi - \frac{\pi}{2}\right)$ for integer n

$c = 2n\pi + \frac{\pi}{2}$

$a = -1, b = 3, c = \left(2n + \frac{1}{2}\right)\pi$

33

$f_1 = f(x)$

Translate $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$:

$f_2(x) = f_1(x-2) = f(x-2)$

Stretch horizontally with scale factor $\frac{1}{5}$:

$g(x) = f_2(5x) = f(5x-2)$

Alternatively, $g(x) = f\left(5\left(x - \frac{2}{5}\right)\right)$

This is a horizontal stretch with scale factor $\frac{1}{5}$ followed by a translation $\begin{pmatrix} 0.4 \\ 0 \end{pmatrix}$

34

$f_1(x) = ax^2 + bx + c$

Translate $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$:

$f_2(x) = f_1(x-2) = a(x-2)^2 + b(x-2) + c = ax^2 + (b-4a)x + c - 2b + 4a$

Stretch horizontally with scale factor 3:

$f_3(x) = f_2\left(\frac{x}{3}\right) = \frac{a}{9}x^2 + \frac{b-4a}{3}x + c - 2b + 4a = x^2 + cx + 14$

Comparing coefficients:

$$\begin{cases} x^2: \frac{a}{9} = 1 & (1) \\ x^1: \frac{b-4a}{3} = c & (2) \\ x^0: c - 2b + 4a = 14 & (3) \end{cases}$$

$(1): a = 9$

$(2): b = 3c + 36$

$(3): c - 2b = -22$

Substituting: $-5c - 72 = -22$

$c = -10$

$b = 6$

Solution: $a = 9, b = 6, c = -10$

35

$f_1(x) = ax^3 + bx + c$

Stretch horizontally with scale factor $\frac{1}{2}$:

$f_2(x) = f_1(2x) = 8ax^3 + 2bx + c$

Translate $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\begin{aligned} f_3(x) = f_2(x+1) &= 8a(x+1)^3 + 2b(x+1) + c \\ &= 8ax^3 + 24ax^2 + (24a+2b)x + 8a + 2b + c \end{aligned}$$

$8ax^3 + 24ax^2 + (24a+2b)x + 8a + 2b + c = 2x^3 + 6x^2 - bx - 2$

Comparing coefficients:

$$\begin{cases} x^3: 8a = 2 & (1) \\ x^2: 24a = 6 & (2) \\ x^1: 24a + 2b = -b & (3) \\ x^0: 8a + 2b + c = -2 & (4) \end{cases}$$

$(1), (2): a = \frac{1}{4}$

$(3): 3b = -6 \Rightarrow b = -2$

$(4): c = 0$

36

$f_1(x) = f(2x + 1)$

Translate $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$: $f_2(x) = f_1(x - 0.5) = f(2(x - 0.5) + 1) = f(2x)$

Horizontal stretch scale factor $\frac{2}{3}$: $f_3(x) = f_2\left(\frac{3}{2}x\right) = f(3x)$

A translation right by half a unit followed by a horizontal stretch scale factor $\frac{2}{3}$ would achieve the result.

Alternatively:

$f_1(x) = f(2x + 1)$

Horizontal stretch scale factor $\frac{2}{3}$: $f_2(x) = f_1\left(\frac{3}{2}x\right) = f(3x + 1)$

Translate $\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$: $f_2(x) = f_1\left(x - \frac{1}{3}\right) = f(3x)$

A horizontal stretch scale factor $\frac{2}{3}$ followed by a translation right by one third of a unit would achieve the same result.

37

$f_1(x) = \tan\left(3x - \frac{\pi}{2}\right)$

Translate $\begin{pmatrix} -\frac{\pi}{6} \\ 0 \end{pmatrix}$: $f_2(x) = f_1\left(x + \frac{\pi}{6}\right) = \tan(3x)$

Reflect through y-axis: $f_3(x) = f_2(-x) = \tan(-3x)$

38

$f_1(x) = 8^x$

Vertical stretch scale factor 5: $f_2(x) = 5f_1(x) = 5 \times 8^x$

$g_1(x) = 2^x$

Translate $\begin{pmatrix} c \\ 0 \end{pmatrix}$: $g_2(x) = g_1(x - c) = 2^{x-c} = 2^{-c} \times 2^x$

Stretch horizontally with scale factor $\frac{1}{3}$: $g_3(x) = g_2(3x) = 2^{-c} \times 2^{3x} = 2^{-c} \times 8^x$

Require that $5 \times 8^x \equiv 2^{-c} \times 8^x$

$5 = 2^{-c}$

$c = -\log_2 5$

Exercise 7E

15

$\mathrm{f}(-x) = \frac{(-x)^3}{(-x)^2 - 6} = -\frac{x^3}{x^2 - 6} = -\mathrm{f}(x)$

$\mathrm{f}(x)$ is an odd function.

16

$\mathrm{f}(-x) = \tan(-x) + 3(-x)^2 = -\tan x + 3x^2$

$\mathrm{f}(x)$ is neither odd nor even.

17

$\mathrm{f}(-x) = (-x)\cos(-x) - \sin(-x) = -x\cos x + \sin x = -\mathrm{f}(x)$

$\mathrm{f}(x)$ is an odd function.

18 **a**

$\mathrm{f}(x) = (x+4)^2 + 3, x \geq k$

This is one-to-one for $k = -4$

b

$y = \mathrm{f}(x) = (x+4)^2 + 3, x \geq -4$ has range $y \geq 3$

$\sqrt{y-3} = x + 4$

$x = -4 + \sqrt{y-3} = \mathrm{f}^{-1}(y)$

Changing variables:

$\mathrm{f}^{-1}(x) = -4 + \sqrt{x-3}$, with domain $x \geq 3$

19 **a**

$\mathrm{f}(x) = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

The vertex of the full function is at $\left(\frac{3}{2}, -\frac{5}{4}\right)$

The function $\mathrm{f}(x)$ can be one-to-one with reduced domain $x \leq \frac{3}{2}$

b

$y = \mathrm{f}(x) = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}, x \leq \frac{3}{2}$ has range $y \geq -\frac{5}{4}$

$\sqrt{y + \frac{5}{4}} = \frac{3}{2} - x$

$x = \frac{3}{2} - \sqrt{y + \frac{5}{4}} = \mathrm{f}^{-1}(y)$

Changing variables:

$\mathrm{f}^{-1}(x) = \frac{3}{2} - \sqrt{x + \frac{5}{4}}$, with domain $x \geq -\frac{5}{4}$

20 **a**

$y = \mathrm{f}(x) = \dfrac{3-2x}{2}$

$2y = 3 - 2x$

$2x = 3 - 2y$

$x = \dfrac{3-2y}{2} = \mathrm{f}^{-1}(y)$

Changing variables:

$\mathrm{f}^{-1}(x) = \dfrac{3-2x}{2} = \mathrm{f}(x)$

Therefore $\mathrm{f}(x)$ is self-inverse.

b

A self-inverse function will have range and domain that match; the domain of $\mathrm{f}^{-1}(x)$ is $x \in \mathbb{R}$

21 **a**

$\mathrm{f}(x)$ is even so $\mathrm{f}(-x) = \mathrm{f}(x)$

$\mathrm{g}(x)$ is odd so $\mathrm{g}(-x) = -\mathrm{g}(x)$

$\mathrm{h}(-x) = \dfrac{\mathrm{f}(-x)}{\mathrm{g}(-x)} = \dfrac{\mathrm{f}(x)}{-\mathrm{g}(x)} = -\mathrm{h}(x)$

Therefore $\mathrm{h}(x)$ is an odd function.

b

The sum of an even function and an odd function is neither even nor odd.

For example:

$\mathrm{f}(x) = 1$ is an even function since $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x

$\mathrm{g}(x) = x$ is an odd function since $\mathrm{g}(-x) = -x = -\mathrm{g}(x)$ for all x

Let $\mathrm{k}(x) = \mathrm{f}(x) + \mathrm{g}(x) = 1 + x$

$\mathrm{k}(1) = 2, \mathrm{k}(-1) = 0$

Then $\mathrm{k}(a) \neq \pm a$ for $a = 1$; therefore $\mathrm{k}(x)$ is neither odd nor even.

22

If $\mathrm{f}(x)$ is even then $\mathrm{f}(-x) = \mathrm{f}(x)$ for all x in the domain of $\mathrm{f}(x)$

If $\mathrm{f}(x)$ is odd then $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all x in the domain of $\mathrm{f}(x)$

If both are true then $\mathrm{f}(x) = -\mathrm{f}(x) \Rightarrow \mathrm{f}(x) = 0$ for all x in the domain of $\mathrm{f}(x)$

23

$\mathrm{f}(x) = |x-1| + |x+1|$

$$\begin{aligned}\text{Then } \mathrm{f}(-x) &= |-x-1| + |-x+1| \\ &= |x+1| + |x-1| \\ &= \mathrm{f}(x)\end{aligned}$$

Therefore $\mathrm{f}(x)$ is even.

24 a

$\text{f}(x) = x^3 + 6x^2 + 9x - 2$ for $-5 \leq x \leq 1$
$\text{f}'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) = 3(x + 1)(x + 3)$
Stationary points are at $x = -1$ and $x = -3$
$\text{f}(-3) = -27 + 54 - 27 - 2 = -2$
$\text{f}(-1) = -1 + 6 - 9 - 2 = -6$
The end point values are $\text{f}(-5) = -22$ and $\text{f}(1) = 14$
Positive cubic with two stationary points will have local maximum at the first and local minimum at the second.
Then the function is one-to-one for $-3 \leq x \leq -1$
It is alternatively one-to-one for $-1 \leq x \leq 1$ or for $-5 \leq x \leq -3$

b

If $\text{f}(x)$ has domain $-3 \leq x \leq -1$ then it has range $-6 \leq \text{f}(x) \leq -2$
$\text{f}^{-1}(x)$ has domain $-6 \leq x \leq -2$

If $\text{f}(x)$ has domain $-5 \leq x \leq -3$ then it has range $-22 \leq \text{f}(x) \leq -6$
$\text{f}^{-1}(x)$ has domain $-22 \leq x \leq -6$

If $\text{f}(x)$ has domain $-1 \leq x \leq 1$ then it has range $-2 \leq \text{f}(x) \leq 14$
$\text{f}^{-1}(x)$ has domain $-2 \leq x \leq 14$

25 a

$\text{f}(x) = x^4 - 8x^2 + 5 = (x^2 - 4)^2 - 11$ for $x \geq k$
$\text{f}'(x) = 4x^3 - 16x = 4x(x^2 - 4)$
Stationary points are at $x = 0, \pm 2$; the positive quartic has two local minima at $x = \pm 2$ and a local maximum at $x = 0$.
The function is one-to-one for $x \geq 2$ so $k = 2$

b

$\text{f}(2) = 16 - 32 + 5 = -11$
The range of $\text{f}(x)$ for $x \geq 2$ is $\text{f}(x) \geq -11$
Therefore, the domain of $\text{f}^{-1}(x)$ is $x \geq -11$

26 a

$\text{f}(x) = \text{e}^x - 4x$ for $x \leq k$
$\text{f}'(x) = \text{e}^x - 4$
There is a single stationary point at $x = \ln 4$, which must be a local minimum since the function grows without limit as $x \to -\infty$
Then the domain over which $\text{f}(x)$ is one-to-one is $x \leq \ln 4$ so $k = \ln 4$

b

$\text{f}(\ln 4) = 4 - 4\ln 4$
The range of $\text{f}(x)$ for $x \leq \ln 4$ is $\text{f}(x) \geq 4(1 - \ln 4)$
Therefore, the domain of $\text{f}^{-1}(x)$ is $x \geq 4(1 - \ln 4)$

27 **a**

Let $y = \mathrm{f}(x) = \dfrac{2x+1}{3x-2}$
$(3x-2)y = 2x+1$
$3xy - 2x = 2y + 1$
$x = \dfrac{2y+1}{3y-2} = \mathrm{f}^{-1}(y)$
Changing variable: $\mathrm{f}^{-1}(x) = \dfrac{2x+1}{3x-2} = \mathrm{f}(x)$
Therefore, $\mathrm{f}(x)$ is self-inverse.

b

The domain of $\mathrm{f}^{-1}(x)$ must be the same as the domain of $\mathrm{f}(x)$ if the function is self-inverse: $x \neq \frac{2}{3}$

28

Let $y = \mathrm{f}(x) = ax + b$
$x = \dfrac{y-b}{a} = \mathrm{f}^{-1}(y)$
Changing variable:
$\mathrm{f}^{-1}(x) = \dfrac{x-b}{a}$
If the function is self-inverse then $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$
$\dfrac{x-b}{a} = ax + b$
$x - b = a^2x + ab$
$x(1-a^2) = b(1+a)$
$(a+1)(1-a)x = b(1+a)$
For the above to be true for all values of x in the domain, one of the following must be true:

- $b = 0$ and the domain is $x = 0$ (which is a trivial case)
- $b = 0, a = 1$: $\mathrm{f}(x) = x$
- $a = -1$: $\mathrm{f}(x) = b - x$

29 **a**

Let $\mathrm{g}(x) = \mathrm{f}(x) + \mathrm{f}(-x)$
Then $\mathrm{g}(-x) = \mathrm{f}(-x) + \mathrm{f}(x) = \mathrm{g}(x)$ so by definition $\mathrm{g}(x)$ is an even function.

b

Let $\mathrm{h}(x) = \mathrm{f}(x) - \mathrm{f}(-x)$
Then $\mathrm{h}(-x) = \mathrm{f}(-x) - \mathrm{f}(x) = -\mathrm{h}(x)$ so by definition $\mathrm{h}(x)$ is an odd function.

c

Since $\mathrm{f}(x) = \frac{1}{2}\mathrm{g}(x) + \frac{1}{2}\mathrm{h}(x)$, it follows that every function can be expressed as the sum of an even and an odd function.

30

Let $y = \mathrm{f}(x) = \dfrac{3-2x}{x+c}$

$(x+c)y = 3-2x$

$x(y+2) = 3-cy$

$x = \dfrac{3-cy}{y+2} = \mathrm{f}^{-1}(y)$

Changing variable:

$\mathrm{f}^{-1}(x) = \dfrac{3-cx}{x+2}$

If $\mathrm{f}(x)$ is self-inverse then $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of $\mathrm{f}(x)$

$\dfrac{3-2x}{x+c} = \dfrac{3-cx}{x+2}$

$(3-2x)(x+2) = (3-cx)(x+c)$

$-2x^2 - x + 6 = -cx^2 + (3-c^2)x + 3c$

$x^2(c-2) + x(c^2-4) + 6 - 3c = 0$ for all x

$(c-2)[x^2 + (c+2)x - 3] = 0$ for all x

This can only be true for $c = 2$, or if $x = \frac{-(c+2)\pm\sqrt{c^2+4c+16}}{2}$

Since the domain is (assumed to be) $x \neq -2$, the only possible solution is that $c = 2$

Mixed Practice

1 **a**

Vertical asymptotes occur at the roots of the denominator

$x = \dfrac{2}{3}$ or -2

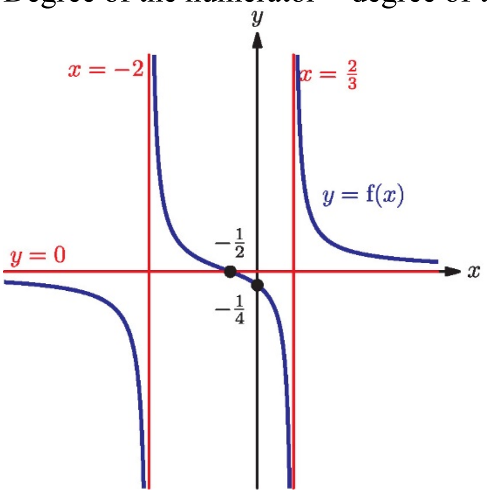
b

When $x = 0$, $y = -\frac{1}{4}$ so the y-intercept is $\left(0, -\frac{1}{4}\right)$

The x-intercept occurs at the root of the numerator: $\left(-\frac{1}{2}, 0\right)$

c

Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



2 **ai**

Vertical asymptotes occur at the roots of the denominator

$x = 4$

aii

As $x \to \infty, y \to x$ so the oblique asymptote is $y = x - 2$

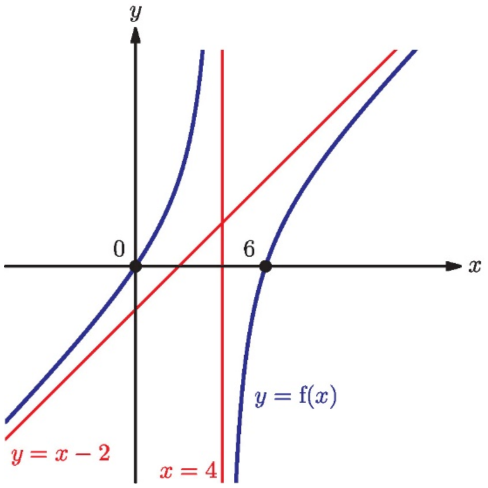
b

$$f(x) = \frac{x^2 - 4x - 2x}{x - 4} = \frac{x(x - 6)}{x - 4}$$

When $x = 0, y = 0$ so the y-intercept is $(0, 0)$

The x-intercepts occur at the roots of the numerator, so there is also an x-intercept at $(6, 0)$

c



3

$6x + x^2 - 2x^3 < 0$

Boundaries are at solutions to $2x^3 - x^2 - 6x = 0$

$x(2x + 3)(x - 2) = 0$

The function is a negative cubic with three distinct roots: $-\frac{3}{2}$, 0 and 2

It will take negative values between the first and second root and for x greater than the third root:

$-\frac{3}{2} < x < 0$ or $x > 2$

4 **a**

Let $f(x) = x^3 - 3x^2 - 6x + 8$

$f(-2) = -8 - 12 + 12 + 8 = 0$ so by the factor theorem, $(x + 2)$ is a factor of $f(x)$

b

$x^3 - 1 \geq 3(x^2 + 2x - 3)$

$x^3 - 3x^2 - 6x + 8 \geq 0$

$(x + 2)(x^2 - 5x + 4) \geq 0$

$(x + 2)(x - 1)(x - 4) \geq 0$

$f(x)$ is a positive cubic with three distinct roots: -2, 1 and 4

It will take positive values between the first and second root and for x greater than the third root:

$-2 \leq x \leq 1$ or $x \geq 4$

5

$2x^4 - 5x^2 + x + 1 < 0$

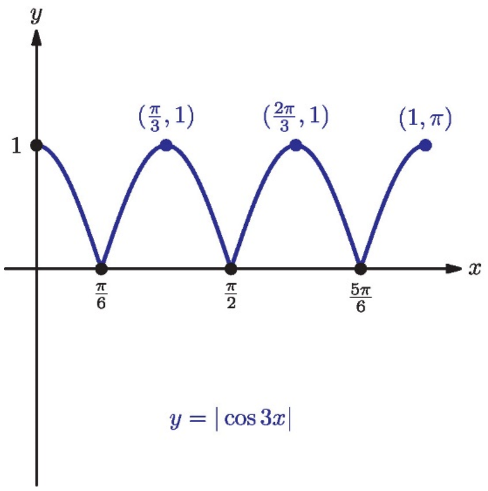
From calculator, $-1.62 \le x \le -0.366$ or $0.618 \le x \le 1.37$

6

$\ln x \le e^{\sin x}$

From calculator, $0 < x \le 3.04$ or $7.01 \le x \le 8.56$

7 **a**



b

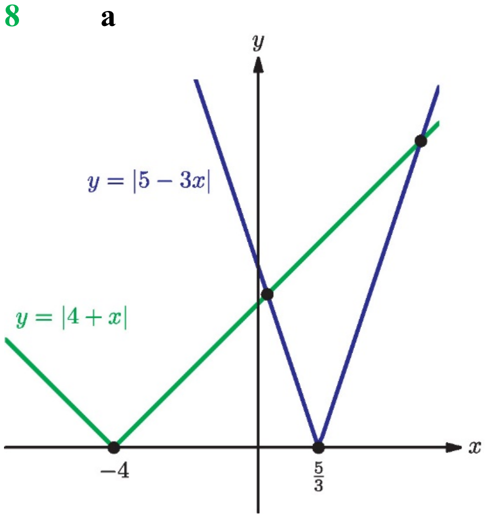
$|\cos 3x| = \frac{1}{2}$ for $0 \le x \le \pi$

$\cos 3x = \pm\frac{1}{2} \Rightarrow 3x = n\pi \pm \frac{\pi}{3}$

$x = \frac{1}{3}n\pi \pm \frac{\pi}{9}$

Within the interval, the solutions are $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

8 **a**



b

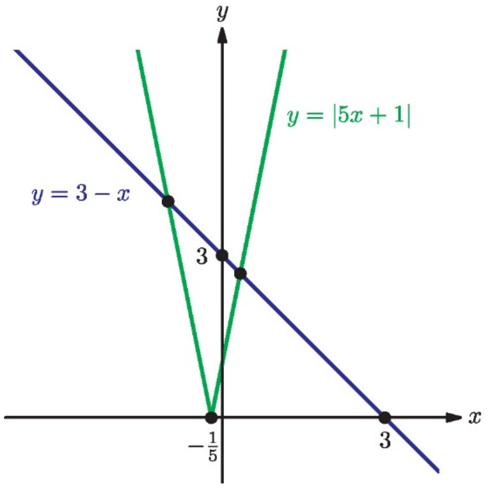
Intersections of the two graphs:

$4 + x = 5 - 3x \Rightarrow x = \frac{1}{4}$

$4 + x = 3x - 5 \Rightarrow x = \frac{9}{2}$

$|4 + x| \le |5 - 3x|$ for $x \le \frac{1}{4}$ or $x \ge \frac{9}{2}$

9 **a**



b

Intersections of the two graphs:

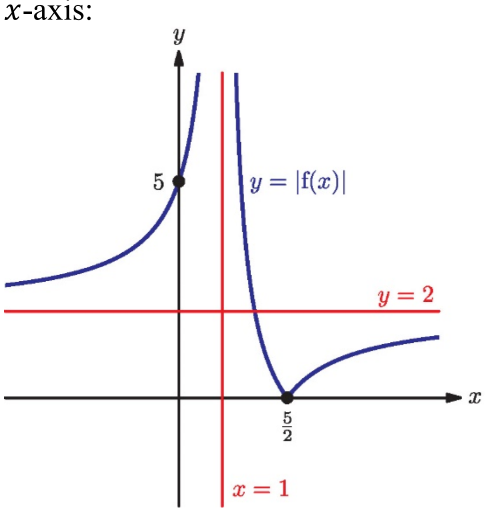
$3 - x = -1 - 5x \Rightarrow x = -1$

$3 - x = 1 + 5x \Rightarrow x = \frac{1}{3}$

$3 - x > |5x + 1|$ for $-1 < x < \frac{1}{3}$

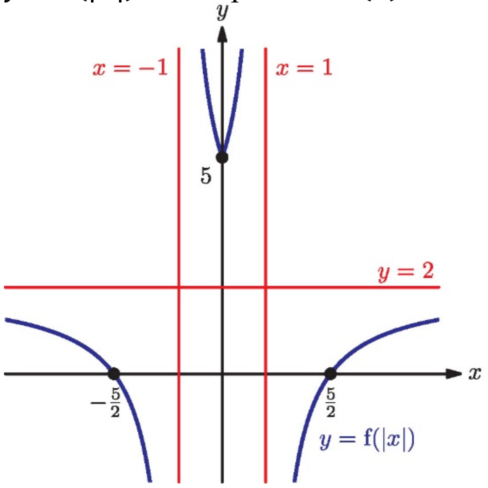
10 **a**

$y = |f(x)|$ takes the parts of $f(x)$ that lie below the x-axis and reflects them through the x-axis:



b

$y = \mathrm{f}(|x|)$ is composed of $\mathrm{f}(x)$ for $x \geq 0$ and its reflection through the y-axis:



11 **a**

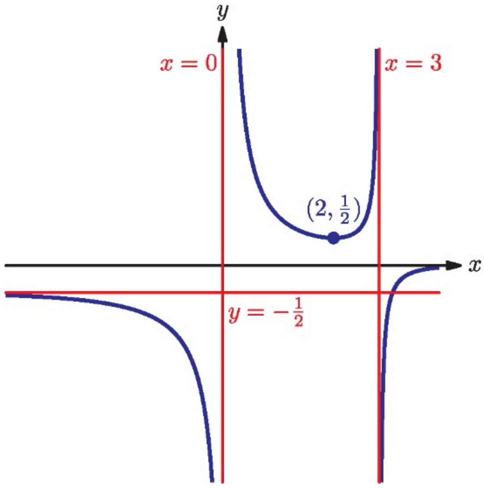
$\frac{1}{\mathrm{f}(x)}$ has vertical asymptotes at roots of $\mathrm{f}(x)$: $x = 0, x = 3$

Local minima of $\mathrm{f}(x)$ are maxima of $\frac{1}{\mathrm{f}(x)}$ and vice versa:

$\frac{1}{\mathrm{f}(x)}$ has min at $\left(2, \frac{1}{2}\right)$

$\mathrm{f}(x) \to -2$ as $x \to -\infty$ so $\frac{1}{\mathrm{f}(x)} \to -\frac{1}{2}$

$\mathrm{f}(x) \to -\infty$ as $x \to \infty$ so $\frac{1}{\mathrm{f}(x)} \to 0$



b

$[\mathrm{f}(x)]^2$ has the same roots as $\mathrm{f}(x)$, intersections with the axis become minima.

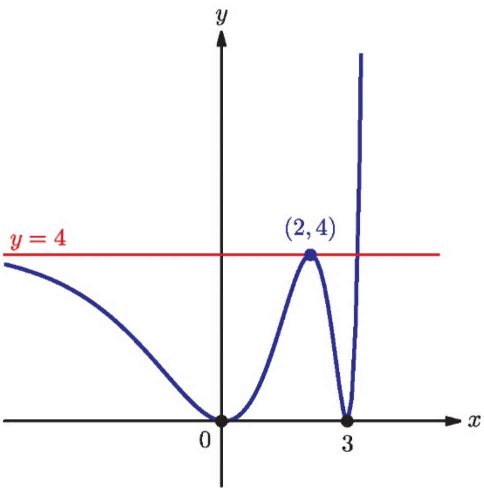
Local minima at $(0, 0), (3, 0)$

Positive stationary points keep the same character, negative stationary points take the opposite character:

$[\mathrm{f}(x)]^2$ has max at $(2, 4)$

$\mathrm{f}(x) \to -2$ as $x \to -\infty$ so $[\mathrm{f}(x)]^2 \to 4$

$\mathrm{f}(x) \to -\infty$ as $x \to \infty$ so $[\mathrm{f}(x)]^2 \to \infty$



c

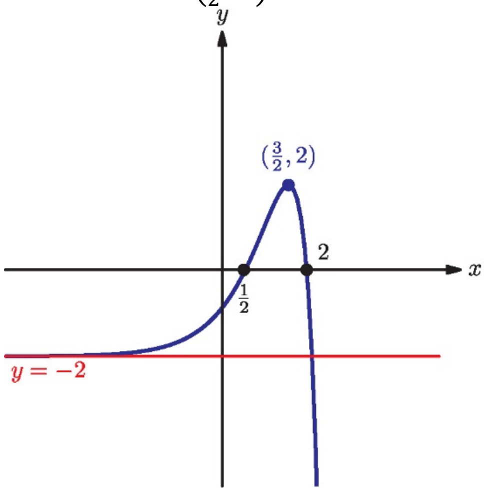
Transforming $y = \mathrm{f}(x)$ to $y = \mathrm{f}(2x-1)$:

$\mathrm{f}_1(x) = \mathrm{f}(x-1)$: translation $\begin{pmatrix}1\\0\end{pmatrix}$

New roots are $(1, 0)$ and $(4, 0)$, max is at $(3, 2)$. Asymptote remains $y = -2$

$\mathrm{f}_2(x) = \mathrm{f}_1(2x)$: Horizontal stretch with scale factor $\frac{1}{2}$

New roots are $\left(\frac{1}{2}, 0\right)$ and $(2, 0)$, max is at $\left(\frac{3}{2}, 2\right)$. Asymptote remains $y = -2$



12

$\mathrm{f}(x) = 3^x + 3^{-x}$

Then $\mathrm{f}(-x) = 3^{-x} + 3^x = \mathrm{f}(x)$

Therefore $\mathrm{f}(x)$ is an even function

13 **a**

$\mathrm{f}(x) = -(x-3)^2 + 5$ so the vertex is at $(3, 5)$

The function is therefore one-to-one for $x \geq 3$

$k = 3$

b

Let $y = \mathrm{f}(x) = -(x-3)^2 + 5, x \geq 3$

$5 - y = (x-3)^2$

$x = 3 + \sqrt{5-y} = \mathrm{f}^{-1}(y)$

Changing variables, $\mathrm{f}^{-1}(x) = 3 + \sqrt{5-x}$

The range of $\mathrm{f}(x)$ is $\mathrm{f}(x) \leq 5$ so the domain of $\mathrm{f}^{-1}(x)$ is $x \leq 5$

14 **a**

$\mathrm{f}(x)$ is even so $\mathrm{f}(-x) = \mathrm{f}(x)$ for all x

$ax^2 + bx + c = a(-x)^2 + b(-x) + c$

$ax^2 + bx + c = ax^2 - bx + c$

$2bx = 0$ for all x

Therefore $b = 0$

b

$\mathrm{g}(x)$ is odd so $\mathrm{g}(-x) + \mathrm{g}(x) = 0$ for all x

$p\sin x + qx + r + p\sin(-x) + q(-x) + r = 0$

$\sin(-x) = -\sin(x)$ so $2r = 0$

Therefore $r = 0$

c

If $\mathrm{h}(x)$ is both odd and even, then $\mathrm{h}(x) = -\mathrm{h}(-x)$ and $\mathrm{h}(-x) = \mathrm{h}(x)$ for all x
Therefore $\mathrm{h}(x) = -h(x)$ for all x
Hence $\mathrm{h}(x) = 0$

15 **ai**

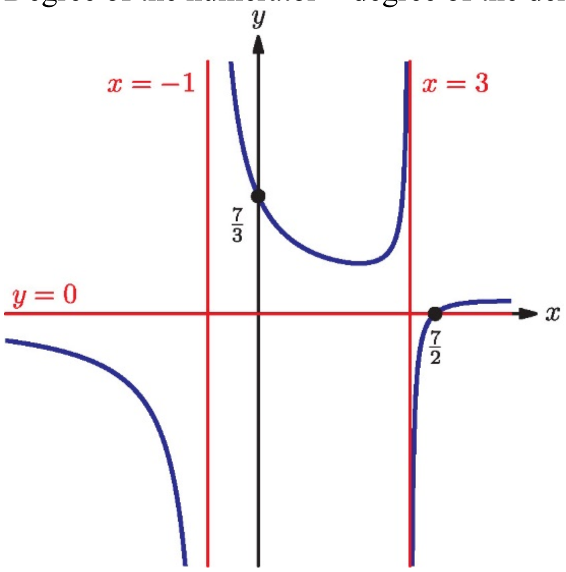
Real roots when discriminant $\Delta \geq 0$
$\Delta = \left(-2(k+1)\right)^2 - 4(k)(7-3k) \geq 0$
$16k^2 - 20k + 4 \geq 0$
$4k^2 - 5k + 1 \geq 0$
$(4k-1)(k-1) \geq 0$
Positive quadratic in k will have values greater than zero outside the roots
$k \leq \frac{1}{4}$ or $k \geq 1$

aii

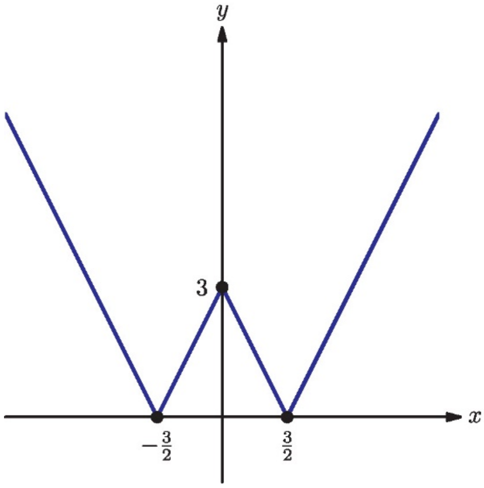
The range of the function $\mathrm{f}(x)$ is the set of values k for which $\mathrm{f}(x) = k$ has real solutions.
$\frac{2x-7}{x^2-2x-3} = k$
$2x - 7 = kx^2 - 2kx - 3k$
$kx^2 - 2(k+1)x + 7 - 3k = 0$
From part **a**, the range of $\mathrm{f}(x)$ is therefore $\mathrm{f}(x) \leq \frac{1}{4}$ or $\mathrm{f}(x) \geq 1$

b

When $x = 0, y = \frac{7}{3}$ so the y-intercept is $\left(0, \frac{7}{3}\right)$
x-intercept at root of the numerator: $\left(\frac{7}{2}, 0\right)$
Vertical asymptotes occur at roots of the denominator: $(x-3)(x+1) = 0$ so $x = -1$ or $x = 3$
Degree of the numerator < degree of the denominator: $y \to 0$ as $x \to \pm\infty$



16 **a**



b

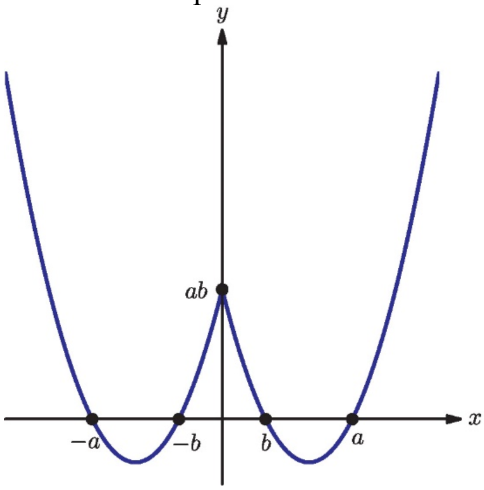
$|2|x|-3|=2$
$2|x|-3=\pm 2$
$2|x|=3\pm 2$
$|x|=1.5\pm 1$
$x=\pm(1.5\pm 1)=\pm 2.5, \pm 0.5$

17

$\mathrm{f}(x)$ is a positive quadratic with roots at a and b.
$y=\mathrm{f}(|x|)$ is composed of $\mathrm{f}(x)$ for $x\geq 0$ and its reflection through the y-axis:

a

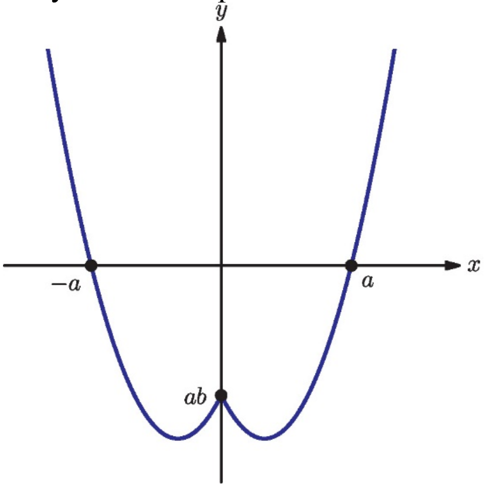
$0<b<a$
Both roots are positive



b

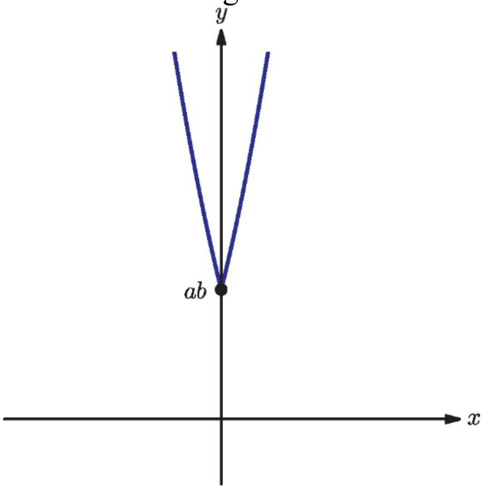
$b < 0 < a$

Only one root is positive



c

Both roots are negative



18 **a**

$f_1(x) = f\left(\frac{x}{3}\right)$: Horizontal stretch with scale factor 3

$f_2(x) = f_1(x - 6) = f\left(\frac{x-6}{3}\right)$: Translation $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$y = f(x)$ is transformed to $y = f\left(\frac{x-6}{3}\right)$ by a horizontal stretch with scale factor 3 followed by a translation 6 units in the positive x direction.

b

Alternatively:

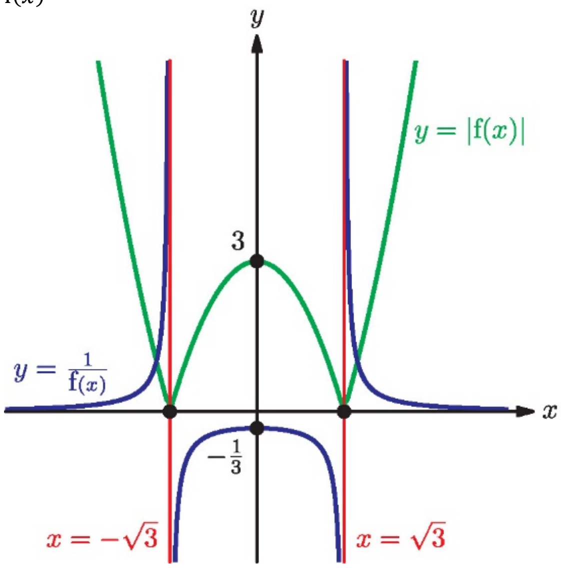
$f_1(x) = f(x - 2)$: Translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$f_2(x) = f_1\left(\frac{x}{3}\right) = f\left(\frac{x}{3} - 2\right) = f\left(\frac{x-6}{3}\right)$: Horizontal stretch with scale factor 3

$y = f(x)$ is transformed to $y = f\left(\frac{x-6}{3}\right)$ by a translation 2 units in the positive x direction followed by a horizontal stretch with scale factor 3.

19 a

$\mathrm{f}(x)$ is a positive quadratic with roots at $\pm\sqrt{3}$.
$y = |\mathrm{f}(x)|$ takes the parts of $\mathrm{f}(x)$ that lie below the x-axis and reflects them through the x-axis
$y = \frac{1}{\mathrm{f}(x)}$ has vertical asymptotes at the roots of $\mathrm{f}(x)$ and since $\mathrm{f}(x) \to \infty$ as $x \to \pm\infty$, $\frac{1}{\mathrm{f}(x)} \to 0$



b

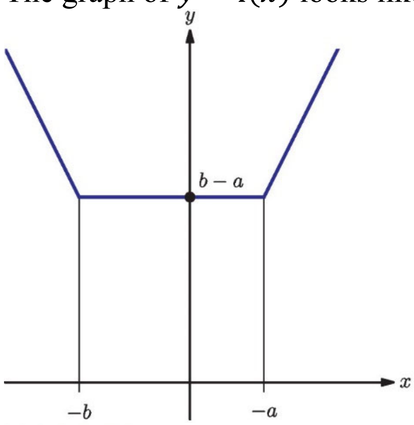
The curves intersect when $|f(x)| = \frac{1}{f(x)}$
$\mathrm{f}(x)|\mathrm{f}(x)| = 1$
$\mathrm{f}(x) = 1$
$x^2 = 4$
$x = \pm 2$
Then $|f(x)| \leq \frac{1}{f(x)}$ for $-2 \leq x < -\sqrt{3}$ or $\sqrt{3} < x \leq 2$

20

$\mathrm{f}(x) = |x + a| + |x + b|$
Assume without loss of generality that $a \leq b$

Then $\mathrm{f}(x) = \begin{cases} -a - b - 2x & x \leq -b \\ b - a & -b < x < -a \\ a + b + 2x & -a < x \end{cases}$

The graph of $y = \mathrm{f}(x)$ looks like this:



$f(x) = |x + a| + |x + b|; a < b$, no y axis shown
Require that $f(x)$ is even so the graph must be symmetrical through the y-axis, line $x = 0$
Since the graph is only symmetrical through $x = -\frac{a}{2} - \frac{b}{2}$ then $a = -b$

You could alternatively use algebraic manipulation or by breaking into cases to show that $a = -b$ is the only condition which allows $f(x)$ to be even, but a graphical approach is tidily direct.

21 **a**
$f(x) = (e^x - 4)^2 - 9$ is a quadratic in e^x, which has its vertex at $e^x = 4$, or $x = \ln 4$
Therefore, restricting the domain so that $f(x)$ is one-to-one, $x \le k$ where $k = \ln 4$

b
Let $y = f(x) = (e^x - 4)^2 - 9, x \le \ln 4$
$e^x - 4 = -\sqrt{y + 9}$
$x = \ln\left(4 - \sqrt{y + 9}\right) = f^{-1}(y)$
Changing variables:
$f^{-1}(x) = \ln\left(4 - \sqrt{x + 9}\right)$
As $x \to -\infty, f(x) \to 7$ so the range of $f(x)$ for $x \le \ln 4$ is $-9 \le f(x) < 7$
Then the domain of $f^{-1}(x)$ is $-9 \le x < 7$

22 **a**
$f(x) = xe^{0.5x}$
$f'(x) = e^{0.5x} + 0.5xe^{0.5x} = (1 + 0.5x)e^{0.5x}$
$f''(x) = 0.5e^{0.5x} + 0.5(1 + 0.5x)e^{0.5x} = (1 + 0.25x)e^{0.5x}$

b
The stationary point of $f(x)$ from part **a** is when $1 + 0.5x = 0$ so $x = -2$
The function is therefore one-to-one if the domain is restricted to $x \ge -2$
$k = -2$

c
As $x \to \infty, f(x) \to \infty$ so the range of $f(x)$ is $-2e^{-1} \le f(x)$
Therefore, the domain of $f^{-1}(x)$ is $x \ge -2e^{-1}$

23 **ai**
$$f(x) = \frac{3x}{x^2 + 1}$$
$$f(-x) = \frac{3(-x)}{(-x)^2 + 1} = -\frac{3x}{x^2 + 1} = -f(x) \text{ for all } x \text{ in the domain of } f(x)$$
Therefore, $f(x)$ is an odd function.

aii
The graph of $f(x)$ will have degree 2 rotational symmetry about the origin.

bi

$f(x) = k$

$kx^2 - 3x + k = 0$

This quadratic has real solutions if discriminant $\Delta \geq 0$

$\Delta = (-3)^2 - 4k^2 = 9 - 4k^2 \geq 0$

$4k^2 - 9 \leq 0$

bii

Therefore, the range of $f(x)$ is $-\frac{3}{2} \leq f(x) \leq \frac{3}{2}$

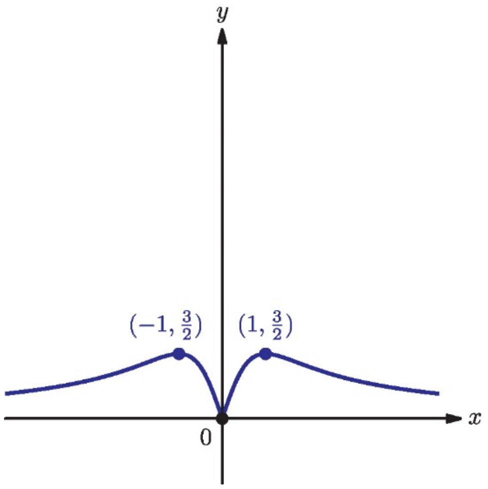
$k = \frac{3}{2}: \frac{3}{2}x^2 - 3x + \frac{3}{2} = 0 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$

$k = -\frac{3}{2}: -\frac{3}{2}x^2 - 3x - \frac{3}{2} = 0 \Rightarrow x^2 + 2x + 1 = 0 \Rightarrow (x+1)^2 = 0 \Rightarrow x = -1$

The turning points are $\left(-1, -\frac{3}{2}\right)$ and $\left(1, \frac{3}{2}\right)$

c

$y = |f(x)|$ takes the parts of $f(x)$ that lie below the x-axis and reflects them through the x-axis



d

Intersections of $y = |x|$ and $y = |f(x)|$:

$f(x) = \pm x$

$\frac{3x}{x^2 + 1} = \pm x$

$x^2 + 1 = \pm 3$

$x^2 = 2$

$x = \pm\sqrt{2}$

Then $|f(x)| \geq |x|$ for $-\sqrt{2} \leq x \leq \sqrt{2}$

24 **a**

x is in the domain of $g(x)$ if $f(x - a) \neq b$

$x - a \neq \pm a$

$x \neq 0, 2a$

b

Transforming $f(x)$ to $g(x) = \frac{1}{f(x-a)-b}$:

$f_1(x) = f(x-a)$: Translate $\begin{pmatrix} a \\ 0 \end{pmatrix}$

Local minimum at $(a, 0)$, passes through $(0, b)$ and $(2a, b)$

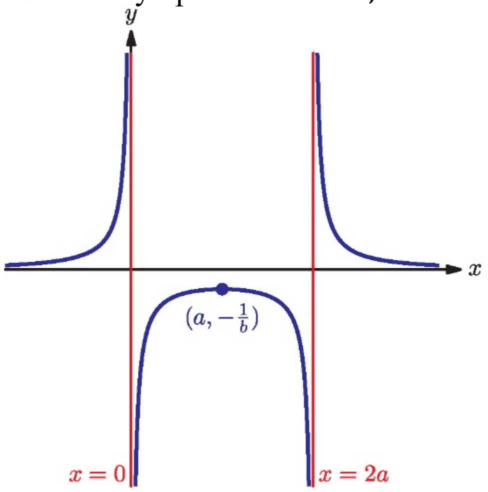
$f_2(x) = f_1(x) - b = f(x-a) - b$: Translate $\begin{pmatrix} 0 \\ -b \end{pmatrix}$

Local minimum at $(a, -b)$, passes through $(0, 0)$ and $(2a, 0)$

$f_3(x) = \frac{1}{f_2(x)} = g(x)$: Reciprocal transformation

Local maximum at $\left(a. -\frac{1}{b}\right)$,

vertical asymptotes at $x = 0, x = 2a$. As $x \to \pm\infty$, $g(x) \to 0$



25 **a**

$$f(x) = \frac{2x-1}{x+2} = \frac{2(x+2)-5}{x+2} = 2 - \frac{5}{x+2}$$

b

$$f'(x) = \frac{5}{(x+2)^2} > 0 \text{ for all } x$$

c

Since from part **b**, $f(x)$ is always increasing, the range is the interval between $f(-1) = -3$ and $f(8) = \frac{3}{2}$

$-3 \leq f(x) \leq 1.5$

di

Let $y = f(x) = \frac{2x-1}{x+2}$

$xy + 2y = 2x - 1$

$x(y-2) = -1 - 2y$

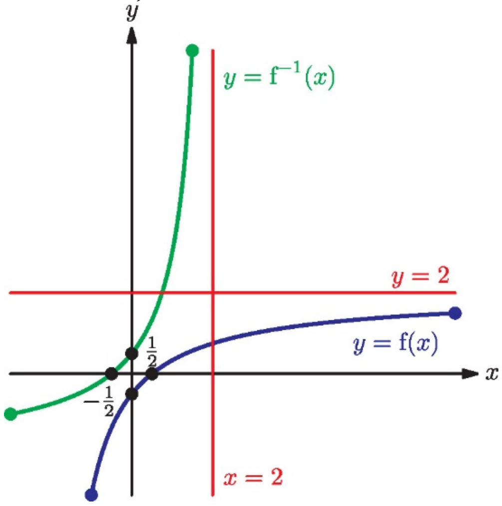
$$x = \frac{1+2y}{2-y} = f^{-1}(y)$$

Changing variables:

$$f^{-1}(x) = \frac{1+2x}{2-x}$$

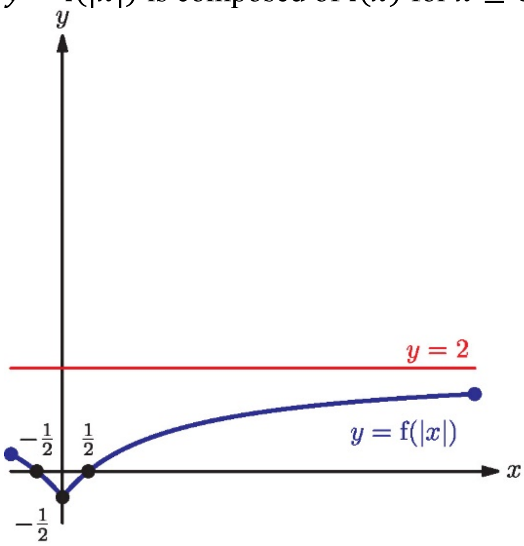
Range of $f(x)$ is $-3 \leq f(x) \leq 1.5$ so the domain of $f^{-1}(x)$ is $-3 \leq x \leq 1.5$

dii, diii



ei

$y = f(|x|)$ is composed of $f(x)$ for $x \geq 0$ and its reflection through the y-axis:



dii

$$f(|x|) = -\frac{1}{4}$$

$$2|x| - 1 = -\frac{1}{4}(|x| + 2)$$

$$\frac{9}{4}|x| = \frac{1}{2}$$

$$|x| = \frac{2}{9}$$

$$x = \pm\frac{2}{9}$$

26 **a**

$$f(x) = \frac{x^2 + 7x + 10}{x + 1} = \frac{x(x + 1) + 6(x + 1) + 4}{x + 1} = x + 6 + \frac{4}{x + 1}$$

As $x \to \infty, f(x) \to x + 6$

The oblique asymptote is $y = x + 6$

b

If $f(x) = k$ has real solutions then $x^2 + 7x + 10 = k(x + 1)$ has real solutions

$x^2 + (7 - k)x + (10 - k) = 0$

This quadratic has real solutions if its discriminant $\Delta \geq 0$

$\Delta = (7 - k)^2 - 4(10 - k)$

$k^2 - 10k + 9 \geq 0$

$(k - 1)(k - 9) \geq 0$

This is a positive quadratic in k, which will have values greater than zero for k outside the roots.

$k \leq 1$ or $k \geq 9$

Therefore, the function $f(x)$ has range $f(x) \leq 1$ or $f(x) \geq 9$

When $k = 1$: $x^2 + 6x + 9 = 0 = (x + 3)^2$ so $x = -3$

When $k = 9$: $x^2 - 2x + 1 = 0 = (x - 1)^2$ so $x = 1$

The turning points are $(-3, 1)$ and $(1, 9)$

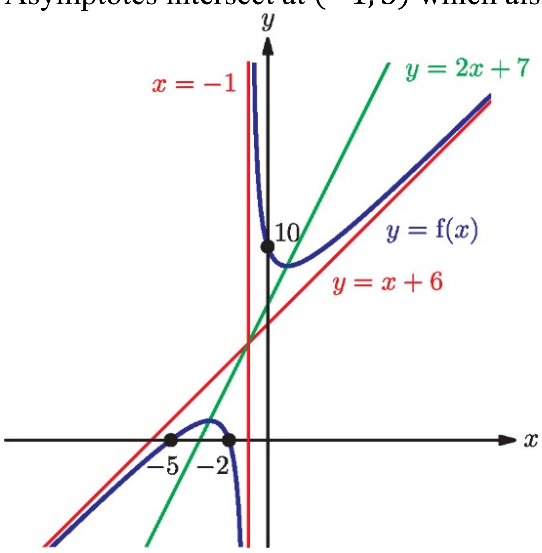
c

When $x = 0, y = 10$ so the y-intercept is $(0, 10)$

x-intercepts are at the roots of the numerator: $(x + 2)(x + 5) = 0$ so $(-5, 0)$ and $(-2, 0)$

Vertical asymptote at the root of the denominator: $x = -1$

Asymptotes intersect at $(-1, 5)$ which also lies on the line $y = x + 6$



d

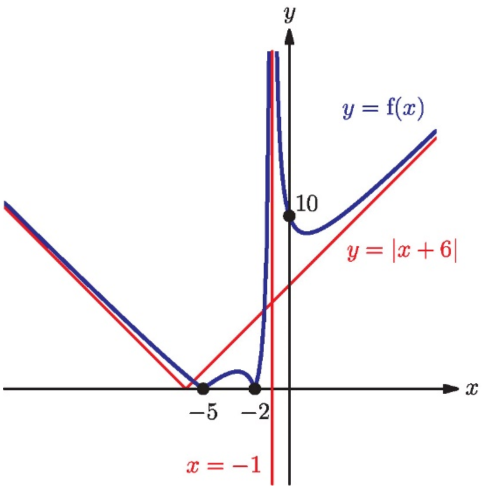
The two graphs intersect where $\frac{x^2+7x+10}{x+1} = 2x + 7$

$x^2 + 7x + 10 = (2x + 7)(x + 1)$
$x^2 + 7x + 10 = 2x^2 + 9x + 7$
$x^2 + 2x - 3 = 0$
$(x + 3)(x - 1) = 0$
$x = -3$ or 1

$$\frac{x^2 + 7x + 10}{x + 1} < 2x + 7 \text{ for } -3 < x < -1 \text{ or } x > 1$$

e

$y = |f(x)|$ takes the parts of $f(x)$ that lie below the x-axis and reflects them through the x-axis



f

For there to be two solutions, c must either equal zero or be a value above the local maximum of the left part of the function and below the local minimum of the right part of the function.
$c = 0$ or $1 < c < 9$

27

$$f(x)=\frac{ax^2+bx+c}{dx+e}, g(x)=\frac{1}{f(x)}=\frac{dx+e}{ax^2+bx+c}$$

Vertical asymptotes of $g(x)$ are the roots of its denominator.

$$ax^2+bx+c=a\left(x-\frac{3}{2}\right)(x+4)=ax^2+\frac{5}{2}ax-6a$$

Comparing coefficients: $b=\frac{5}{2}a, c=-6a$

$f(x)=\frac{ax^2+bx+c}{dx+e}$ has oblique asymptote $y=x+1$ so $ax^2+bx+c=(x+1)(dx+e)+k$ for some remainder k.

$$\frac{a}{2}(2x^2+5x-12)=dx^2+(d+e)x+e+k$$

Comparing coefficients: $d=a, d+e=\frac{5}{2}a$ so $e=\frac{3}{2}a$

Putting all these details together,

$$f(x)=\frac{a\left(x^2+\frac{5}{2}x-6\right)}{a\left(x+\frac{3}{2}\right)}=\frac{2x^2+5x-12}{2x+3}$$

If $f(x)=g(x)$ then $[f(x)]^2=1$ so $f(x)=\pm 1$

$2x^2+5x-12=\pm(2x+3)$

$2x^2+3x-15=0$ or $2x^2+7x-9=0$

$2x^2+3x-15=0$ or $(2x+9)(x-1)=0$

$$x=\frac{-3\pm\sqrt{129}}{4} \text{ or } x=-\frac{9}{2} \text{ or } 1$$

$x=2.09, -3.59, -4.5$ or 1

28

This is a question equivalent to the final question in exercise 7E.

Rather than again find the inverse and equate the functions, a different approach is shown below for variety. Students may consider which they find more elegant.

$$f(x)=\frac{3x-5}{x+c}, x\neq -c$$

If $f(x)$ is self-inverse then $f(f(x))=x$ for all x in the domain.

$$\frac{3\left(\frac{3x-5}{x+c}\right)-5}{\frac{3x-5}{x+c}+c}=x$$

$$\frac{3(3x-5)-5(x+c)}{3x-5+c(x+c)}=x$$

$4x-15-5c=(3+c)x^2+(c^2-5)x$

$(3+c)x^2+(c^2-9)x+15+5c=0$

$(3+c)[x^2+(c-3)x+5]=0$

For this to be true for all x in the domain, $c=-3$

8 Vectors

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 8A

28 **a** $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \mathbf{a} + \frac{1}{2}\mathbf{b}$

b $\overrightarrow{EF} = \overrightarrow{EC} + \overrightarrow{CF} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

c $\overrightarrow{DG} = \overrightarrow{DA} + \overrightarrow{AG} = -\mathbf{b} + 2\mathbf{a}$

29 **a** $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \mathbf{a} - \mathbf{b}$

b $\overrightarrow{ON} = \overrightarrow{OB} + \frac{1}{3}\overrightarrow{BA} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

c $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = -\frac{1}{2}\mathbf{a} + \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{6}(4\mathbf{b} - \mathbf{a})$

30 **a** $3\mathbf{a} - \mathbf{c} + 5\mathbf{b} = 3\begin{pmatrix}2\\-1\\3\end{pmatrix} - \begin{pmatrix}-2\\2\\3\end{pmatrix} + 5\begin{pmatrix}0\\1\\-5\end{pmatrix} = \begin{pmatrix}8\\0\\-19\end{pmatrix}$

b $|\mathbf{b} - 2\mathbf{a}| = \left|\begin{pmatrix}-4\\3\\-11\end{pmatrix}\right| = \sqrt{16 + 9 + 121} = \sqrt{146}$

31

$\left|\begin{pmatrix}3k\\-k\\k\end{pmatrix}\right| = |k|\sqrt{9 + 1 + 1} = \sqrt{11}|k| = 22$

$|k| = 2\sqrt{11}$

$k = \pm 2\sqrt{11}$

32

$\left|\begin{pmatrix}2\\3t\\t-1\end{pmatrix}\right| = 3 = \sqrt{4 + 9t^2 + (t-1)^2}$

$4 + 9t^2 + t^2 - 2t + 1 = 9$

$10t^2 - 2t - 4 = 0$

$t = \dfrac{1 \pm \sqrt{41}}{10}$

33

$\mathbf{x} = \frac{1}{4}(\mathbf{b} - 3\mathbf{a}) = \frac{1}{4}\left(\begin{pmatrix}5\\3\\3\end{pmatrix} - 3\begin{pmatrix}-1\\1\\2\end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix}8\\0\\-3\end{pmatrix}$

34

$t\mathbf{b} = t\begin{pmatrix}1\\-1\\2\end{pmatrix} = \mathbf{c} - \mathbf{a} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}3\\-2\\5\end{pmatrix} = \begin{pmatrix}-2\\2\\-4\end{pmatrix} = -2\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$t = -2$

35 **a**

$\frac{1}{\sqrt{36+36+9}} = \frac{1}{9}$

Unit vector is $\frac{1}{9}\begin{pmatrix}6\\6\\-3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

b

$\left|\begin{pmatrix}4\\-1\\2\sqrt{2}\end{pmatrix}\right| = \sqrt{16+1+8} = 5$

A vector of magnitude 10 parallel to the vector is $\begin{pmatrix}8\\-2\\4\sqrt{2}\end{pmatrix}$

36

$\mathbf{a}+p\mathbf{b} = \begin{pmatrix}2+3p\\p\\2+3p\end{pmatrix} = k\begin{pmatrix}3\\2\\3\end{pmatrix}$

Require $p = 2k$ and $2+3p = 3k$

Solving by substitution: $2+6k = 3k$

$k = -\frac{2}{3}$

Then $p = -\frac{4}{3}$

37

$\lambda\mathbf{x}+\mathbf{y} = \begin{pmatrix}2\lambda+4\\3\lambda+1\\\lambda+2\end{pmatrix} = k\begin{pmatrix}0\\1\\0\end{pmatrix}$

Require $\lambda+2=0$

$\lambda = -2$

38

$\left|\begin{pmatrix}4\\-1\\1\end{pmatrix}\right| = \sqrt{16+1+1} = 3\sqrt{2}$

Then $\sqrt{2}\begin{pmatrix}4\\-1\\1\end{pmatrix}$ has magnitude 6

39

$|\mathbf{a}+\lambda\mathbf{b}| = \left|\begin{pmatrix}2\lambda-2\\-\lambda\\2\lambda-1\end{pmatrix}\right| = \sqrt{4\lambda^2-8\lambda+4+\lambda^2+4\lambda^2-4\lambda+1} = 5\sqrt{2}$

$9\lambda^2-12\lambda+5 = 50$

$9\lambda^2-12\lambda-45 = 0$

$3\lambda^2-4\lambda-15 = 0$

$(3\lambda+5)(\lambda-3) = 0$

$\lambda = 3 \text{ or } -\frac{5}{3}$

40

$$\left|\begin{pmatrix} 2t \\ t+3 \\ -2t-1 \end{pmatrix}\right| = \sqrt{4t^2 + t^2 + 6t + 9 + 4t^2 + 4t + 1}$$
$$= \sqrt{9t^2 + 10t + 10}$$
$$= \sqrt{\left(3t + \frac{5}{3}\right)^2 + \frac{65}{9}}$$

From the completed square form, this has minimum value $\frac{\sqrt{65}}{3}$.

41 a

$$\left|\begin{pmatrix} 3\sin\theta \\ -3\cos\theta \\ 4 \end{pmatrix}\right| = \sqrt{9\sin^2\theta + 9\cos^2\theta + 16} = \sqrt{25} = 5, \text{independent of the value of } \theta$$

b

$$\left|\begin{pmatrix} 1+3\sin\theta \\ 1-3\cos\theta \\ 4 \end{pmatrix}\right| = \sqrt{2 + 6(\sin\theta - \cos\theta) + 25}$$
$$= \sqrt{(27 + 6(sin\theta\ - cos\theta))}$$

$$\sin\theta - \cos\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right) = \sqrt{2}\left(\sin\left(\theta - \frac{\pi}{4}\right)\right)$$

So this has maximum value $\sqrt{2}$

Then $\left|\begin{pmatrix} 1+3\sin\theta \\ 1-3\cos\theta \\ 4 \end{pmatrix}\right|$ has minimum value $\sqrt{27 + 6\sqrt{2}}$

This can be easily interpreted geometrically:

The position vector describes the locus of a circle parallel to the x-y plane, radius 3 and centred at (0,0,4)

The value found is the distance between $(1, 1, 0)$ and the point on the circle furthest from it, which will by rapid consideration be at $(-3\sqrt{2}, -3\sqrt{2}, 4)$.

The distance between them is $\sqrt{27 + 6\sqrt{2}}$.

Exercise 8B

16

$$AB = \left|\begin{matrix}-2\\-2\\1\end{matrix}\right| = 3$$

Then distance $AC = \frac{3}{2}$

17 **a**

$$\overrightarrow{AB} = \begin{pmatrix}3\\-5\\1\end{pmatrix}, \overrightarrow{CA} = \begin{pmatrix}6\\-10\\2\end{pmatrix} = 2\overrightarrow{AB}$$

Therefore the three points are collinear.

b

$AB: BC = 1:3$

18 **a**

$$AB = |\mathbf{b} - \mathbf{a}| = \left|\begin{pmatrix}1\\-2\\5\end{pmatrix}\right| = \sqrt{1+4+25} = \sqrt{30}$$

b

$$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$$

19 **a**

$$\overrightarrow{BC} = \begin{pmatrix}4\\-2\\-10\end{pmatrix} \text{ so } \overrightarrow{BD} = \begin{pmatrix}2\\-1\\-5\end{pmatrix} \text{ and so } D \text{ has position vector } \mathbf{b} + \overrightarrow{BD} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$$

b

$$\overrightarrow{AD} = \begin{pmatrix}-3\\13\\-7\end{pmatrix} \text{ so } AD = \sqrt{9+169+49} = \sqrt{227}$$

20

$$\mathbf{d} = (2\mathbf{i} - 3\mathbf{j}) + (\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - 4\mathbf{j}$$

21

$$\overrightarrow{AB} = \begin{pmatrix}-1\\12\\-3\end{pmatrix} = \overrightarrow{BC} \text{ so } \mathbf{c} = \mathbf{b} + \overrightarrow{BC} = \begin{pmatrix}2\\13\\-1\end{pmatrix}$$

22 **a**

$$\text{Require } \overrightarrow{CD} = \overrightarrow{BA} = \begin{pmatrix}2\\-2\\-1\end{pmatrix} \text{ so } \mathbf{d} = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

b

$AB = \sqrt{4+4+1} = 3$

$BC = \left|\begin{pmatrix}-2\\3\\0\end{pmatrix}\right| = \sqrt{4+9} = \sqrt{13}$. Since adjacent sides have unequal length, parallelogram $ABCD$ is not a rhombus.

23 **a**

Require $\overrightarrow{CD} = \overrightarrow{BA} = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$ so D has coordinates $(13, 4, -6)$

b

If M is the midpoint of AC then $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}\begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix}$ so M has coordinates $(6, 3, 1)$

c

If N is the midpoint of BD then $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\begin{pmatrix} 14 \\ 2 \\ -14 \end{pmatrix}$ so N has coordinates $(6, 3, 1)$

M and N are the same point; the midpoint of AC is also the midpoint of BD (the centre of the parallelogram).

24 **a**

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = \frac{1}{2}(\mathbf{a} - \mathbf{b}) + \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{b})$

b

BC and MN must be parallel, with MN half the length of BC, since $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$

25

If M is the midpoint of AB, N is the midpoint of BC and so on then:

$\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}), \mathbf{n} = \frac{1}{2}(\mathbf{b} + \mathbf{c}), \mathbf{p} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, and $\mathbf{q} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$,

Then the vectors joining consecutive vertices of $MNPQ$ are

$\overrightarrow{MN} = \mathbf{n} - \mathbf{m} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{NP} = \mathbf{p} - \mathbf{n} = \frac{1}{2}(\mathbf{d} - \mathbf{b})$

$\overrightarrow{QP} = \mathbf{p} - \mathbf{q} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{MQ} = \mathbf{q} - \mathbf{m} = \frac{1}{2}(\mathbf{d} - \mathbf{b})$

Since $\overrightarrow{MN} = \overrightarrow{QP}$ and $\overrightarrow{NP}$

$= \overrightarrow{MQ}$, it follows by definition that $MNPQ$ is a parallelogram.

26 **a**

$\overrightarrow{AC} = \begin{pmatrix} 15 \\ 5 \\ -5 \end{pmatrix} = k\overrightarrow{AB} = k\begin{pmatrix} 3 + p \\ q - 2 \\ -3 \end{pmatrix}$

Inspecting the third component, $k = \frac{3}{5}$

So $3 + p = 9 \Rightarrow p = 6$

And $q - 2 = 3 \Rightarrow q = 5$

b

$AB : BC = k : (1 - k) = 3 : 2$

27

$\overrightarrow{AB} = \begin{pmatrix} -2 \\ -5 \\ 4 \end{pmatrix}$

$\overrightarrow{AC} = \frac{2}{5}\overrightarrow{AB}$ so C has position vector $\mathbf{a} + \begin{pmatrix} -0.8 \\ -2 \\ 1.6 \end{pmatrix} = 2.2\mathbf{i} - \mathbf{j} - 2.4\mathbf{k}$

28

$\overrightarrow{AB} = \begin{pmatrix} 2+2t \\ 4+t \\ -9-5t \end{pmatrix}$ so $AB = \sqrt{4 + 8t + 4t^2 + 16 + 8t + t^2 + 81 + 90t + 25t^2} = 3$

Squaring: $30t^2 + 106t + 101 = 9$

$15t^2 + 53t + 46 = 0$

$(t+2)(15t+23) = 0$

$t = -2 \text{ or } -\frac{23}{15}$

29 **a**

$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{q}) = \frac{1}{2}(3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

b

Require $\overrightarrow{QR} = \overrightarrow{MQ} = \frac{1}{2}(-\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

R has coordinates $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

30 **a**

D has position vector $\mathbf{d} = \mathbf{b} + \frac{1}{3}(\mathbf{c} - \mathbf{b}) = \frac{1}{3}(2\mathbf{b} + \mathbf{c})$

M has position $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{d}) = \frac{1}{6}(3\mathbf{a} + 2\mathbf{b} + \mathbf{c})$

b

X has position $\mathbf{x} = \mathbf{a} + \frac{1}{4}(\mathbf{c} - \mathbf{a}) = \frac{1}{4}(3\mathbf{a} + \mathbf{c})$

$\overrightarrow{BM} = \mathbf{m} - \mathbf{b} = \frac{1}{6}(3\mathbf{a} - 4\mathbf{b} + \mathbf{c})$

$\overrightarrow{MX} = \mathbf{x} - \mathbf{m} = \frac{1}{12}\big(9\mathbf{a} + 3\mathbf{c} - 2(3\mathbf{a} + 2\mathbf{b} + \mathbf{c})\big) = \frac{1}{12}(3\mathbf{a} - 4\mathbf{b} + \mathbf{c}) = \frac{1}{2}\overrightarrow{BM}$

This shows that B, M and X are collinear, with $BM : MX = 2 : 1$

31 **a**

$\mathbf{d} = \mathbf{b} + \frac{2}{5}(\mathbf{c} - \mathbf{b}) = \frac{1}{5}(3\mathbf{b} + 2\mathbf{c})$

$\mathbf{e} = \mathbf{a} + \frac{1}{2}(\mathbf{a} - \mathbf{c}) = \frac{1}{2}(3\mathbf{a} - \mathbf{c})$

$\mathbf{f} = \mathbf{b} + \frac{2}{3}(\mathbf{a} - \mathbf{b}) = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$

b

$\overrightarrow{DF} = \mathbf{f} - \mathbf{d} = \frac{1}{15}(10\mathbf{a} - 4\mathbf{b} - 6\mathbf{c})$

$\overrightarrow{FE} = \mathbf{e} - \mathbf{f} = \frac{1}{6}(5\mathbf{a} - 2\mathbf{b} - 3\mathbf{c}) = \frac{1}{12}(10\mathbf{a} - 4\mathbf{b} - 6\mathbf{c}) = \frac{5}{4}\overrightarrow{DF}$

This shows that D, F and E are collinear, with $DF : FE = 4 : 5$

32 **a**

$\overrightarrow{AB} = \begin{pmatrix} k-2 \\ 2+k \end{pmatrix} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \begin{pmatrix} k-3 \\ 2k+1 \end{pmatrix} = \overrightarrow{AD}$

None of these vectors can be the zero vector so, by definition, $ABCD$ is a parallelogram.

b

$AB = \sqrt{(k-2)^2 + (2+k)^2} = \sqrt{2k^2+8}$

$BC = \sqrt{(k-3)^2 + (2k+1)^2} = \sqrt{5k^2 - 2k + 10}$

For $ABCD$ to be a rhombus, these two side lengths must be equal

$2k^2 + 8 = 5k^2 - 2k + 10$

$3k^2 - 2k + 2 = 0$

The discriminant for this quadratic is $(-2)^2 - 4(3)(2) = -20 < 0$ so there are no real roots.

There is no real value k for which the adjacent side lengths are equal, so $ABCD$ cannot be a rhombus.

Exercise 8C

19 **a**

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} = 8 + 8 + 3 = 19$$

b

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{c}) = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0 + 6 + 1 = 7$$

c

$$(\mathbf{b} + \mathbf{d}) \cdot (2\mathbf{a}) = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = 16 + 8 + 8 = 32$$

20

$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$$

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{OA}}{|AB||OA|} = \frac{(-6 + 10 - 3)}{\sqrt{35}\sqrt{17}} = \frac{1}{\sqrt{595}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{595}}\right) = 87.7°$$

21

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \overrightarrow{BD} = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$$

$$\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|AC||BD|} = \frac{24 + 0 - 1}{\sqrt{17}\sqrt{53}} = \frac{23}{\sqrt{901}}$$

$$\theta = \cos^{-1}\left(\frac{23}{\sqrt{901}}\right) = 40.0°$$

22

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{-1+1+4}{\sqrt{6}\sqrt{6}} = \frac{4}{6} = \frac{2}{3}$$

23

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{10}{15} = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

24

$$\cos\theta = \frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|} = \frac{-15}{9\times 12} = -\frac{5}{36}$$

$$\theta = \cos^{-1}\left(-\frac{5}{36}\right) = 98.0°$$

25 **a**

$$\mathbf{b}\cdot\mathbf{d} = \begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\1\end{pmatrix} = (1\times 0) + \big(3\times(-2)\big) + (-1\times 1) = -7$$

$$\mathbf{d}\cdot\mathbf{b} = \begin{pmatrix}0\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-1\end{pmatrix} = (0\times 1) + \big((-2)\times 3\big) + \big(1\times(-1)\big) = -7$$

Scalar product of vectors is commutative because it can be represented as the sum of products of elements, and arithmetic multiplication is commutative.

b

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}6\\3\\-4\end{pmatrix} = 2\times 6 + 1\times 3 + (-2)\times(-4) = 23$$

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} &= \begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-1\end{pmatrix} + \begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}5\\0\\-3\end{pmatrix}\\ &= [2\times 1 + 1\times 3 + (-2)\times(-1)] + [2\times 5 + 1\times 0 + (-2)\times(-3)]\\ &= 7 + 16 = 23\end{aligned}$$

Scalar product of vectors is distributive across addition because it can be represented as the sum of products of elements, and arithmetic multiplication is distributive across addition.

c

$$(\mathbf{c}-\mathbf{d})\cdot\mathbf{c} = \begin{pmatrix}5\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}5\\0\\-3\end{pmatrix} = 5\times 5 + 2\times 0 + (-4)\times(-3) = 37$$

$$\begin{aligned}|\mathbf{c}|^2 - \mathbf{c}\cdot\mathbf{d} &= \left|\begin{pmatrix}5\\0\\-3\end{pmatrix}\right|^2 - \begin{pmatrix}5\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\1\end{pmatrix}\\ &= (5^2 + 0^2 + (-3)^2) - [5\times 0 + 0\times(-2) + (-3)\times 1]\\ &= 34 + 3 = 37\end{aligned}$$

d

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \begin{pmatrix}3\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-3\end{pmatrix} = 3\times3+4\times4+(-3)\times(-3)=34$$

$$|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b} = \left|\begin{pmatrix}2\\1\\-2\end{pmatrix}\right|^2 + \left|\begin{pmatrix}1\\3\\-1\end{pmatrix}\right|^2 + 2\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-1\end{pmatrix}$$
$$= (2^2+1^2+(-2)^2)+(1^2+3^2+(-1)^2)$$
$$+2\big(2\times1+1\times3+(-2)\times(-1)\big)$$
$$= 9+11+2(7)=34$$

26

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$\frac{1}{2}=\frac{12}{8|\mathbf{b}|}$$
$$|\mathbf{b}|=3$$

27

Position vectors of the vertices are $\mathbf{a}=\begin{pmatrix}1\\1\\3\end{pmatrix}, \mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{c}=\begin{pmatrix}5\\1\\2\end{pmatrix}$

So $\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}, \mathbf{c}-\mathbf{a}=\begin{pmatrix}4\\0\\-1\end{pmatrix}, \mathbf{b}-\mathbf{c}=\begin{pmatrix}-3\\-2\\-1\end{pmatrix}$

$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{4+0+2}{\sqrt{9}\sqrt{17}} = \frac{2}{\sqrt{17}} \Rightarrow A = 61.0°$$
$$\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-3+4+2}{\sqrt{9}\sqrt{14}} = \frac{1}{\sqrt{14}} \Rightarrow B = 74.5°$$
$$C = 180° - A - B = 44.5°$$

28

$$\mathbf{b}-\mathbf{a}=\begin{pmatrix}2\\-2\\3\end{pmatrix}, \mathbf{c}-\mathbf{a}=\begin{pmatrix}5\\0\\-4\end{pmatrix}, \mathbf{b}-\mathbf{c}=\begin{pmatrix}-3\\-2\\7\end{pmatrix}$$
$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{10+0-12}{\sqrt{17}\sqrt{41}} = -\frac{2}{\sqrt{697}} \Rightarrow A = 94.3°$$
$$\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-6+4+21}{\sqrt{17}\sqrt{62}} = \frac{19}{\sqrt{1054}} \Rightarrow B = 54.2°$$
$$C = 180° - A - B = 31.5°$$

29 **a**

$$\mathbf{a}=\begin{pmatrix}1\\-2\\2\end{pmatrix}, \mathbf{b}=\begin{pmatrix}3\\-1\\7\end{pmatrix}, \mathbf{c}=\begin{pmatrix}5\\0\\0\end{pmatrix}$$
$$\mathbf{b}-\mathbf{a}=\begin{pmatrix}2\\1\\5\end{pmatrix}, \mathbf{c}-\mathbf{a}=\begin{pmatrix}4\\2\\-2\end{pmatrix}, \mathbf{b}-\mathbf{c}=\begin{pmatrix}-2\\-1\\7\end{pmatrix}$$
$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{8+2-10}{\sqrt{30}\sqrt{24}} = 0 \Rightarrow A = 90°$$

b

$$\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-4-1+35}{\sqrt{30}\sqrt{54}} = \frac{\sqrt{5}}{3} \Rightarrow B = 41.8°$$

$C = 180° - A - B = 48.2°$

c

$$\text{Area} = \frac{1}{2}(AB)(AC) = \frac{1}{2}\sqrt{30}\sqrt{24} = 6\sqrt{5}$$

30

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$$

$$\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{1\times|\mathbf{q}|}$$

$|\mathbf{q}| = 6$

31

Require $\begin{pmatrix}4+2t\\-1+t\\2+t\end{pmatrix}\cdot\begin{pmatrix}3\\5\\1\end{pmatrix} = 0$

$12 + 6t - 5 + 5t + 2 + t = 0$

$12t = -9$

$t = -\frac{3}{4}$

32

Require $\begin{pmatrix}t\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}2t\\1\\t\end{pmatrix} = 0$

$2t^2 - 3t = 0$

$t = 0, \frac{3}{2}$

33

$\mathbf{a}$ is a unit vector so $\mathbf{a}\cdot\mathbf{a} = 1$

$\mathbf{a}$ is perpendicular to $\mathbf{b}$ so $\mathbf{a}\cdot\mathbf{b} = 0$

$\mathbf{a}\cdot(2\mathbf{a} - 3\mathbf{b}) = 2\mathbf{a}\cdot\mathbf{a} - 3\mathbf{a}\cdot\mathbf{b} = 2$

34 **a**

$\overrightarrow{BC} = \begin{pmatrix}-6-2\lambda\\-17-\lambda\\5\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}-7\\4\\2\end{pmatrix}$

Require $\overrightarrow{BC}\cdot\overrightarrow{AC} = 0$

$42 + 14\lambda - 68 - 4\lambda + 10 = 0$

$10\lambda = 16$

$\lambda = 1.6$

b

From part **a**, $C = 90°$

$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 2.2 \\ 22.6 \\ -3 \end{pmatrix}, \mathbf{c} - \mathbf{a} = \begin{pmatrix} -7 \\ 4 \\ 2 \end{pmatrix}$

$\cos A = \frac{(\mathbf{b}-\mathbf{a}) \cdot (\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{-15.4 + 90.4 - 6}{\sqrt{524.6}\sqrt{69}} = \frac{\sqrt{69}}{\sqrt{524.6}} \Rightarrow A = 68.7°$

$B = 180° - A - C = 21.3°$

c

$\overrightarrow{BC} = \begin{pmatrix} -9.2 \\ -18.6 \\ 5 \end{pmatrix}$ so $BC = \sqrt{455.6}$. From part **b**, $BC = \sqrt{69}$

Area $= \frac{1}{2}(BC)(AC) = \frac{1}{2}\sqrt{455.6}\sqrt{69} = 88.7$

35

a and **b** have equal magnitude: $|\mathbf{a}| = |\mathbf{b}|$ so $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b}$ $(*)$

$(3\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - 3\mathbf{b}) = 0$

Expanding and rearranging:

$\Rightarrow 3\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} - 9\mathbf{a} \cdot \mathbf{b} - 3\mathbf{b} \cdot \mathbf{b} = \mathbf{0}$

$\Rightarrow 3(\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}) - 8\mathbf{a} \cdot \mathbf{b} = 0$

$\Rightarrow 8\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ by $(*)$

$\Rightarrow \mathbf{a} \cdot \mathbf{b} = \mathbf{0}$

$\Rightarrow$ **a** is perpendicular to **b**

36 **a**

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \overrightarrow{AD} - \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

b

$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a}$

$= \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}$

$= |\mathbf{b}|^2 - |\mathbf{a}|^2$

c

If $ABCD$ is a rhombus then $|\mathbf{a}| = |\mathbf{b}|$ so $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = 0$

That is, $AC \perp BD$, the diagonals are perpendicular.

37 **a**

If A and B have position vectors **a** and **b** respectively, then $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b} = \lambda\mathbf{a}$

By definition, if $\overrightarrow{OB} = \lambda\overrightarrow{OA}$ then O, A and B are collinear.

b

$\overrightarrow{BA} = (1-\lambda)\begin{pmatrix}2\\1\\4\end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix}2\lambda-12\\\lambda-2\\4\lambda-4\end{pmatrix}$

Require $\overrightarrow{BA}\cdot\overrightarrow{CB} = 0$

$(1-\lambda)(4\lambda-24+\lambda-2+16\lambda-16) = 0$

$(1-\lambda)(21\lambda-42) = 0$

$\lambda = 2$

($\lambda = 1$ represents the degenerate case where A and B are collocated, which does not represent a solution to the problem posed).

c)

B is the point on extended line OA for which the distance from C to the line is shortest.

$\overrightarrow{CB} = \begin{pmatrix}-8\\0\\4\end{pmatrix}$ so $CB = \sqrt{80} = 4\sqrt{5}$

Exercise 8D

26 **a**

$\mathbf{a} = \begin{pmatrix}3\\-1\\5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

Line direction vector $\mathbf{d} = \mathbf{b}-\mathbf{a} = \begin{pmatrix}-4\\2\\-3\end{pmatrix}$

Line has equation $\mathbf{r} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\2\\-3\end{pmatrix}$

b

If $(0, 1, 5)$ lies on the line then $\begin{pmatrix}0\\1\\5\end{pmatrix} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\2\\-3\end{pmatrix}$ for some λ

From the y element: $1 = 2\lambda - 1$ so $\lambda = 1$

This does not provide a consistent solution for the other two elements.

$(0, 1, 5)$ does not lie on the line.

27

The required line has vector equation $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-3\end{pmatrix}$

$\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$; rearranging each of the three element equations to solve for λ gives

$\lambda = \frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-2}{-3}$

The Cartesian equation is $\frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-2}{-3}$

28

The lines have direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

Acute angle θ between the lines is such that

$\cos\theta = \left|\frac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}\right| = \left|\frac{-2}{\sqrt{11}\sqrt{18}}\right|$

$\theta = 81.8°$

29

The lines have direction vectors $\mathbf{d_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\mathbf{d_1} \cdot \mathbf{d_2} = 4 - 1 - 3 = 0$ so the two lines have perpendicular vectors.
Both lines pass through the same point $(0, 0, 1)$ so they are perpendicular within a single plane.

Remember to show or observe that the two vectors are coplanar. While it might be argued that any two lines with perpendicular direction vectors should be considered perpendicular, general usage would say that if the lines do not intersect, they are simply skew and **not** truly perpendicular.

Any pair of parallel lines are necessarily coplanar, and it is standard to also require that two lines must cross at 90° angle (that is, must intersect and so be coplanar) to be considered perpendicular.

30

Substituting $x = 3, y = -2, z = 2$:

$$\begin{cases} \frac{x+1}{2} = 2 \\ \frac{4-y}{3} = 2 \\ \frac{2z}{3} = \frac{4}{3} \neq 2 \end{cases}$$

The point $(3, -2, 2)$ does not lie on the given line.

31 **a**

$|\mathbf{v}| = \sqrt{0.5^2 + 2^2 + 1.5^2} = \sqrt{6.5} = 2.55 \text{ m s}^{-1}$

b

$\mathbf{r} = (12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}) + t(0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$

c

Solving for $\mathbf{r} = (16\mathbf{i} + 8\mathbf{j} + 14\mathbf{k})$:

$$\begin{cases} 12 + 0.5t = 16 & (1) \\ -5 + 2t = 8 & (2) \\ 11 + 1.5t = 14 & (3) \end{cases}$$

$4(1) - (2): 53 = 56$

This is a contradiction, so the particle does not pass through the point $(16, 8, 14)$

32

At $t = 3, \mathbf{r_1} = \begin{pmatrix} 13 \\ 2 \\ 9 \end{pmatrix}$ and $\mathbf{r_2} = \begin{pmatrix} 0.5 \\ 6 \\ -1.5 \end{pmatrix}$

Then $|\mathbf{r_1} - \mathbf{r_2}| = \sqrt{12.5^2 + (-4)^2 + 10.5^2} = \sqrt{282.5} \approx 16.8 \text{ m}$

33 **a**

$x = 0$ so $\lambda = 3$. The point is $(0, 12, 5)$ so $p = 12, q = 5$

b

Direction vector of the line is $\mathbf{d} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

Angle with $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is θ where

$\cos\theta = \frac{\mathbf{d} \cdot \mathbf{k}}{|\mathbf{d}|} = \frac{1}{\sqrt{18}}$

$\theta = 76.4°$

34 **a**

When $t = -2, \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} = \overrightarrow{OA}$ so A lies on the line.

When $t = 0, \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \overrightarrow{OB}$ so B lies on the line.

b

C must then be represented by $t = 2$ so that the distance between AB and BC is the same.

C has coordinates $(0, 3, 0)$

35 **a**

$\overrightarrow{PQ} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$ so the line has vector equation $\mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b

Q is represented by $\lambda = 1$ and P by $\lambda = 0$

Then if $PR = 3PQ$, R must be represented by $\lambda = 3$ or $\lambda = -3$

R has coordinates $(-5, -5, 11)$ or $(19, 7, -7)$

36 **a**

Direction vector $\mathbf{d} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ so the line equation is $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

b

$\left|\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right| = \sqrt{2^2 + (-3)^2 + 6^2} = 7$

c

$AP = 35 = 5|\mathbf{d}|$ so P has position given by $\lambda = \pm 5$

P has coordinates $(12, -14, 34)$ or $(-8, 16, -26)$

37 **a**

Solving for λ:

$\lambda = \frac{x-1}{3} = \frac{y-4}{-2} = \frac{z+1}{3}$

The Cartesian equation is $\frac{x-1}{3} = \frac{y-4}{-2} = \frac{z+1}{3}$

b

Reading off the denominators, $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

$|\mathbf{d}| = \sqrt{22}$ so the unit direction vector is $\frac{1}{\sqrt{22}}\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

38 **a**

$\frac{(3-x)}{2} = \frac{(3z+1)}{4} = \lambda, y = -1$

$x = 3 - 2\lambda, y = -1, z = \frac{4}{3}\lambda - \frac{1}{3}$

$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -\frac{1}{3} \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$

(using $\mu = 1.5(\lambda - 1)$ to get integer values throughout)

b

When $\mu = -\frac{1}{3}, \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ \frac{1}{3} \end{pmatrix}$ so $p = \frac{1}{3}$

39 **a**

$\frac{(2x-1)}{3} = \frac{2-z}{4} = \lambda, y = 7$

Rearranging:

$x = 1.5\lambda + 0.5, y = 7, z = 2 - 4\lambda$

$\mathbf{r} = \begin{pmatrix} 0.5 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1.5 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -8 \end{pmatrix}$

(using $\mu = 2(\lambda - 1)$ to get integer values throughout)

b

Direction vector of the line is $\mathbf{d} = \begin{pmatrix} 3 \\ 0 \\ -8 \end{pmatrix}$

Angle with $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is θ where

$\cos\theta = \frac{\mathbf{d} \cdot \mathbf{i}}{|\mathbf{d}|} = \frac{3}{\sqrt{73}}$

$\theta = 69.4°$

40

Line l_1 has equation $\frac{x-3}{5}=\frac{y-2}{1}=\frac{z-1.5}{-1}$

Line l_2 has equation $\frac{x+1}{3}=\frac{z-3}{-1}, y=1$

Reading direction vectors from the denominators of the Cartesian form:

Direction vectors $\mathbf{d_1}=\begin{pmatrix}5\\1\\-1\end{pmatrix}$ and $\mathbf{d_2}=\begin{pmatrix}3\\0\\-1\end{pmatrix}$

Angle between the lines is given by θ where $\cos\theta=\frac{\mathbf{d_1}\cdot\mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}=\frac{16}{\sqrt{27}\sqrt{10}}$

$\theta=13.2°$

41

C has position vector $\mathbf{c}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ for some value λ

Require that $\overrightarrow{PC}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}=0$

$\begin{pmatrix}2\lambda-3\\-\lambda\\2\lambda-4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}=0$

$9\lambda-14=0$

$\lambda=\frac{14}{9}$

C has coordinates $\left(\frac{64}{9},\frac{4}{9},\frac{19}{9}\right)$

42 **a**

At $t=0$, the position of the object is $(3,-1,4)$

b

Speed is $\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right|=\sqrt{11}\approx 3.32\text{ m s}^{-1}$

c

$\mathbf{r}(3)=\begin{pmatrix}6\\-4\\13\end{pmatrix}$ so the distance from the origin is $\sqrt{6^2+(-4)^2+13^2}=\sqrt{221}$

$\approx 14.9\text{ m}$

43 **a**

$\mathbf{r_1}=(3-2t)\mathbf{i}+(5t)\mathbf{j}$

$\mathbf{r_2}=(4t)\mathbf{i}+(5+t)\mathbf{j}$

b

$\mathbf{r_1}-\mathbf{r_2}=\begin{pmatrix}3-6t\\4t-5\end{pmatrix}$ so $|\mathbf{r_1}-\mathbf{r_2}|=\sqrt{(3-6t)^2+(4t-5)^2}$

$=\sqrt{52t^2-76t+34}$

c

Completing the square:

$$|\mathbf{r_1} - \mathbf{r_2}| = \sqrt{52t^2 - 76t + 34}$$
$$= \sqrt{52\left(t - \frac{38}{52}\right)^2 - \frac{38^2}{52} + 34}$$
$$= \sqrt{52\left(t - \frac{19}{26}\right)^2 + \frac{81}{13}}$$

The minimum separation distance is therefore $\sqrt{\frac{81}{13}} \approx 2.50$

44

Direction vector is $\mathbf{d} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$|\mathbf{d}| = \sqrt{9} = 3$ so the vector position of the aeroplane at time t is $\mathbf{r}$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\left(\frac{894}{3}\right)\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 596 \\ -596 \\ 298 \end{pmatrix}$$

45 **a**

The direction vector of the line $\mathbf{d} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$

$$\overrightarrow{PM} = \begin{pmatrix} 2\lambda - 16 \\ -3\lambda - 4 \\ 3\lambda - 8 \end{pmatrix}$$

Require $\overrightarrow{PM} \cdot \mathbf{d} = 0$

$22\lambda - 44 = 0$

$\lambda = 2$

M has coordinates $(9, -5, 8)$

b

When $\lambda = 5, \mathbf{r} = (15, -14, 17) = \overrightarrow{OQ}$ so Q lies on the line.

c

$\overrightarrow{MQ} = \mathbf{3d}$ so if R is distinct from Q, $\overrightarrow{MR} = -3\mathbf{d}$, so R is at the position described by $\lambda = -1$

R has coordinates $(3, 4, -1)$

46 **a**

When $\lambda = \frac{5}{6}, \mathbf{r}_1 = \frac{1}{6}\begin{pmatrix} 5 \\ 19 \\ 27 \end{pmatrix}$ so P lies on l_1

When $t = \frac{7}{6}, \mathbf{r}_2 = \frac{1}{6}\begin{pmatrix} 5 \\ 19 \\ 27 \end{pmatrix}$ so P also lies on l_2

b

The direction vectors of the lines are $\mathbf{d_1} = \begin{pmatrix}1\\5\\3\end{pmatrix}, \mathbf{d_2} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$

The acute angle θ between the lines has

$\cos\theta = \left|\frac{\mathbf{d_1}\cdot\mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}\right| = \frac{|13|}{\sqrt{35}\sqrt{11}}$

$\theta = 48.5°$

c

When $t = 3, \mathbf{r}_2 = \begin{pmatrix}-1\\5\\10\end{pmatrix}$ so Q lies on l_2

d

$\overrightarrow{PQ} = \frac{11}{6}\mathbf{d_2}$ so $PQ = \frac{11}{6}\sqrt{11} \approx 6.08$

e

If the point on l_1 closest to Q is R then $Q\hat{R}P = 90°$, so $QR = PQ\sin\theta \approx 4.55$

47

Let P, with position vector $\mathbf{r}$, be the point on the line closest to the origin.

Then $\overrightarrow{OP}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix} = 0$

$\left(\begin{pmatrix}1\\-2\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\2\\1\end{pmatrix} = 0$

$0 + 9\lambda = 0$

$\lambda = 0$

$\overrightarrow{OP} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$ so $OP = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

48

Let P be the point with position vector $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and Q the point on the line with position vector $\mathbf{r}$, which lies closest to P.

Then $\overrightarrow{PQ}\cdot\begin{pmatrix}-3\\1\\1\end{pmatrix} = 0$

$\left(\begin{pmatrix}2\\-1\\0\end{pmatrix} + t\begin{pmatrix}-3\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}-3\\1\\1\end{pmatrix} = 0$

$-7 + 11t = 0$

$t = \frac{7}{11}$

$\overrightarrow{PQ} = \frac{1}{11}\begin{pmatrix}1\\-4\\7\end{pmatrix}$ so $PQ = \frac{1}{11}\sqrt{1^2 + (-4)^2 + 7^2} = \frac{\sqrt{66}}{11} \approx 0.739$

Exercise 8E

9

Line AB has equation $\mathbf{r_1} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\5\\5\end{pmatrix}$

Line CD has equation $\mathbf{r_2} = \begin{pmatrix}8\\3\\3\end{pmatrix} + \mu\begin{pmatrix}-2\\0\\0\end{pmatrix}$

Solving $\mathbf{r_1} = \mathbf{r_2}$:

$$\begin{cases} 1 + 5\lambda = 8 - 2\mu & (1) \\ 5\lambda = 3 & (2) \end{cases}$$

$(2): \lambda = \frac{3}{5}$

Point of intersection is $(4, 3, 3)$

10

Direction vectors $\mathbf{d_1} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix}, \mathbf{d_2} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.
Solving for a point of intersection:

$$\begin{cases} -3 - \lambda = 8 - \mu & (1) \\ 5 - 2\lambda = 5 - 3\mu & (2) \\ 2 - 4\lambda = 1 + 3\mu & (3) \end{cases}$$

$(2): 2\lambda = 3\mu$

Substituting into $(3): 2 - 6\mu = 1 + 3\mu \Rightarrow 9\mu = 1$

Then $\mu = \frac{1}{9}, \lambda = \frac{1}{6}$

Substituting into $(1): -3 - \lambda = -\frac{19}{6}, 8 - \mu = \frac{71}{9}$

Since these values are inconsistent with equation (1), the two lines do not intersect; since they are not parallel, they are skew.

When you have completed Section F on the vector product, you will find a more direct way to solve this type of problem.

If the two lines are given by $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{d_1}$ and $\mathbf{r_2} = \mathbf{a_2} + \mu\mathbf{d_2}$:

Find vector $\mathbf{v}$ which is perpendicular to both $\mathbf{d_1}$ and $\mathbf{d_2}$.

Then the value of scalar product $\frac{((\mathbf{a_1} - \mathbf{a_2}) \cdot \mathbf{v})}{|\mathbf{v}|}$ is the shortest distance between the two lines.

In this question, you can determine that $\mathbf{v} = \begin{pmatrix}-18\\7\\1\end{pmatrix}$ is perpendicular to both.

$$\left(\begin{pmatrix}-3\\5\\2\end{pmatrix} - \begin{pmatrix}8\\5\\1\end{pmatrix}\right) \cdot \begin{pmatrix}-18\\7\\1\end{pmatrix} = \begin{pmatrix}-11\\0\\1\end{pmatrix} \cdot \begin{pmatrix}-18\\7\\1\end{pmatrix} = 199$$

$|\mathbf{v}| = \sqrt{374}$ so the minimum distance between the two lines is $\frac{199}{\sqrt{374}}$.

Since this is non-zero, the two lines do not intersect and so are skew.

11 **a**

For l_1: $\begin{cases} x = 2z - 3 \\ y = 2z - 9 \end{cases}$

For l_2: $\begin{cases} x = \dfrac{11 - 3z}{5} \\ y = \dfrac{z - 27}{5} \end{cases}$

b

If the two lines intersect then equating the equations for x in the two lines:

$2z - 3 = \dfrac{11 - 3z}{5} \Rightarrow 10z - 15 = 11 - 3z \Rightarrow z = 2$

Equating the equations for y in the two lines:

$2z - 9 = \dfrac{z - 27}{5} \Rightarrow 10z - 45 = z - 27 \Rightarrow z = 2$

Since the two equations give consistent results, the point, which has coordinates $(1, -5, 2)$, must lie on both lines.

12

Direction vectors $\mathbf{d_1} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.

Solving for a point of intersection:

$$\begin{cases} 1 - \lambda = 7 + 2\mu & (1) \\ \lambda = 2 + 2\mu & (2) \\ 5 + 2\lambda = 7 + 3\mu & (3) \end{cases}$$

$(1) + (2)$: $1 = 9 + 4\mu \Rightarrow \mu = -2$

$(2) \Rightarrow \lambda = -2$

Substituting into (3): $5 - 4 = 7 - 6$

Since this is true, the two lines intersect; the point of intersection is $(3, -2, 1)$

13 **a**

The direction vector of the first line is $\mathbf{d_1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and, reading the denominators of the Cartesian form of the second line, $\mathbf{d_2} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = 3\mathbf{d_1}$

The two direction vectors are parallel.

Substituting $x = -1, y = 1, z = 2$, the position of the known point on the first line, into the second equation, gives

$$\frac{-1 - 5}{6} = \frac{1 - 7}{6} = \frac{2 - 5}{3}$$

Each of these fractions has the value -1, so this is a true statement.

The two lines are parallel and pass through the point $(-1, 1, 2)$ so must be the same line.

b

Substituting $x = 4t - 5, y = 4t - 3, z = 1 + 2t$ into the Cartesian equation in part **a**:

$$\frac{4t}{6} = \frac{4t - 10}{6} = \frac{2t - 4}{3}$$

These equations are inconsistent so there can be no common point between the lines; they are parallel and distinct, not coincident.

14 **a**

The y-axis is the line where $x = z = 0$.

Substituting into the line equation: $\frac{-6}{2} = \frac{y+1}{7} = \frac{9}{-3}$

$y = -22$

The point of intersection with the y-axis is $(0, -22, 0)$

b

The z-axis is the line where $x = y = 0$.

Substituting into the line equation: $\frac{-6}{2} = \frac{1}{7} = \frac{z+9}{-3}$

$\frac{-6}{2} \neq \frac{1}{7}$ so there is no solution where this equation is true.

Hence the line does not intersect the z-axis

15 **a**

The first equation can be parameterised to $x = 3\mu + 2, y = 4\mu - 1, z = \mu - 1$

Substituting these into the second equation:

$3 - 3\mu = \frac{4\mu + 1}{-3} = \frac{\mu - 8}{2}$

$18 - 18\mu = -8\mu - 2 = 3\mu - 24$

$\mu = 2$ gives a consistent solution to this set of equations.

There is a point of intersection at $(8, 7, 1)$

b

If $\lambda = 1$ then the vector equation $\mathbf{r} = \begin{pmatrix} 8 \\ 7 \\ 1 \end{pmatrix}$ which shows that the line passes through the point found in part **a**.

16

$\mathbf{r_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}$

Setting these equal:

$$\begin{cases} \mu = 1 + \lambda & (1) \\ 1 + 2\mu = 7 & (2) \\ \mu - 1 = \lambda p - 4 & (3) \end{cases}$$

$(2): \mu = 3$

$(1): \lambda = 2$

$(3): 2 = 2p - 4$

$p = 3$ allows the two lines to have a consistent solution (point of intersection)

The point of intersection is $(3, 7, 2)$

17 **a**

Direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.

Solving for a point of intersection:

$$\begin{cases} 1 + \lambda = 1 + 2\mu & (1) \\ -\lambda = -1 - \mu & (2) \\ 3 + 2\lambda = 4 + 3\mu & (3) \end{cases}$$

$(1) + (2): 1 = \mu$ so $\lambda = 2$

Substituting into $(3): 7 = 7$

This is true, so the two lines have a point of intersection at $(3, -2, 7)$

b

Although they both pass through the same point, the first line does so at $t = 2$ and the second at $t = 1$ so (taking the two helicopters as having point locations – that is, assuming they are small enough given the scale of the model) they will not collide.

18 **a**

Re-expressing the two particle positions with different time parameters λ and μ:

$$\mathbf{r_1} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 9 \\ -1 \\ 22 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$v_1 = \left| \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \right| = \sqrt{54}$$

$$v_2 = \left| \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right| = \sqrt{9} = 3$$

b

Solving for a point of intersection:

$$\begin{cases} 1 + 2\lambda = 9 - 2\mu & (1) \\ -3 + \lambda = -1 + \mu & (2) \\ 3 + 7\lambda = 22 + 2\mu & (3) \end{cases}$$

$(1) + (3): 4 + 9\lambda = 31 \Rightarrow \lambda = 3, \mu = 1$

Substituting into $(2): -3 + 3 = -2 + 2$ is true, so the system is consistent and the two paths intersect, at point $(7, 0, 24)$.

However, the two particles are not at the intersection point at the same time; the first particle is at the intersection point at $t = 3$ and the second particle is there at $t = 1$, so they do not meet.

19 **a**

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 0.7 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1.2 \\ 0.8 \\ -0.1 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 7.7 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0.3 \end{pmatrix}$$

If one fly is directly above the other then the x and y components must match.

$$\begin{cases} 1.2t = 7.7 - t (1) \\ 0.7 + 0.8t = t (2) \end{cases}$$

$(1): 2.2t = 7.7 \Rightarrow t = 3.5$

$(2): 0.7 = 0.2t \Rightarrow t = 3.5$

The equations are consistent; the two flies are aligned over the x-y plane at $t = 3.5$.

b

$\mathbf{r_1}(3.5) = \begin{pmatrix} 4.2 \\ 3.5 \\ 2.65 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 4.2 \\ 3.5 \\ 2.05 \end{pmatrix}$

The first fly is 0.6 m above the second.

20 **a**

$\mathbf{r_B} = t\begin{pmatrix} 64 \\ 0 \\ 0 \end{pmatrix}, \mathbf{r_S} = \begin{pmatrix} 0 \\ 0.5 \\ -0.02 \end{pmatrix} + kt\begin{pmatrix} 40 \\ -25 \\ c \end{pmatrix}$

If they two vessels coincide at a time t then

$$\begin{cases} 64t = 40kt & (1) \\ 0 = 0.5 - 25kt & (2) \\ 0 = -0.02 + ckt & (3) \end{cases}$$

$(1): k = 1.6$

$(2): 0.5 = 40t \Rightarrow t = 0.0125$

$(3): c = \frac{0.02}{kt} = 1$

b

The velocity of the submarine is $k\begin{pmatrix} 40 \\ -25 \\ c \end{pmatrix} = \begin{pmatrix} 64 \\ -40 \\ 1.6 \end{pmatrix}$ so the speed is 75.5 km h^{-1}

21 **a**

$\overrightarrow{PQ} = \left(\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right) - \left(\begin{pmatrix} 1 \\ -10 \\ 12 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}\right) = \begin{pmatrix} 3 - \lambda + \mu \\ 11 - 3\lambda \\ -7 + 4\lambda + 2\mu \end{pmatrix}$

Require $\overrightarrow{PQ}$ is perpendicular to both lines, so $\overrightarrow{PQ} \cdot \mathbf{d_1} = \overrightarrow{PQ} \cdot \mathbf{d_2} = 0$

$\overrightarrow{PQ} \cdot \mathbf{d_1} = 0$:

$\begin{pmatrix} 3 - \lambda + \mu \\ 11 - 3\lambda \\ -7 + 4\lambda + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = 64 - 26\lambda - 7\mu = 0$

$\Rightarrow 26\lambda + 7\mu = 64 \qquad (1)$

$\overrightarrow{PQ} \cdot \mathbf{d_2} = 0$:

$\begin{pmatrix} 3 - \lambda + \mu \\ 11 - 3\lambda \\ -7 + 4\lambda + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -11 + 7\lambda + 5\mu = 0$

$\Rightarrow 7\lambda + 5\mu = 11 \qquad (2)$

b

$5(1) - 7(2): 81\lambda = 243$

$\lambda = 3$

$\Rightarrow \mu = -2$

$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

$PQ = \sqrt{9} = 3$ is the shortest distance between the two lines.

Exercise 8F

14 **a**

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta = 17.5$

15

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

$7 = 2 \times 5 \sin\theta$

$\theta = \sin^{-1}(0.7) \approx 0.775$

16

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

$$\left|\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}\right| = 7 \times 1 \sin\theta$$

$\sqrt{17} = 7 \sin\theta$

$$\theta = \sin^{-1}\left(\frac{\sqrt{17}}{7}\right) \approx 0.630$$

17

$$|\mathbf{p} \times \mathbf{q}| = \left|\begin{pmatrix} -1 \\ -10 \\ 7 \end{pmatrix}\right| \Rightarrow |\mathbf{p} \times \mathbf{q}| = \sqrt{150} = 5\sqrt{6}$$

18

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ 1 \end{pmatrix}$$

A vector perpendicular to both **a** and **b** is $-8\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

19 **a**

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

b

$$\left|\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{2}$$ so a unit vector perpendicular to the two vectors would be $\frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

20

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$$\left|\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}\right| = \sqrt{14}$$

A unit vector perpendicular to both $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ is $\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

21 **a**

$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = \mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}(\text{distributing})$$
$$= \mathbf{0}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{b}-\mathbf{0}(\mathbf{v}\times\mathbf{v}=\mathbf{0},\mathbf{v}\times\mathbf{w}=-\mathbf{w}\times\mathbf{v})$$
$$= 2\mathbf{a}\times\mathbf{b}$$

b

$$(2\mathbf{a}-3\mathbf{b})\times(3\mathbf{a}+2\mathbf{b}) = 6\mathbf{a}\times\mathbf{a}+4\mathbf{a}\times\mathbf{b}-9\mathbf{b}\times\mathbf{a}-6\mathbf{b}\times\mathbf{b}$$
$$= 13\mathbf{a}\times\mathbf{b}$$

22 **a**

$\mathbf{a}\times\mathbf{b}$ gives a vector perpendicular to both **a** and **b**.
The scalar product of two perpendicular vectors is always zero.
Taking the scalar product of $\mathbf{a}\times\mathbf{b}$ and **a** must therefore produce the value zero.
b)
Distributing:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$$
$$= 0-0(\text{using the reasoning in part } \mathbf{a})$$
$$= 0$$

23 **a**

$$\overrightarrow{AB}=\begin{pmatrix}4\\12\\1\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}-4\\6\\2\end{pmatrix}, \overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}=\begin{pmatrix}-8\\-6\\1\end{pmatrix}$$
$$\mathbf{p}=\begin{pmatrix}4\\12\\1\end{pmatrix}\times\begin{pmatrix}-4\\6\\2\end{pmatrix}=\begin{pmatrix}18\\-12\\72\end{pmatrix}$$
$$\mathbf{q}=\begin{pmatrix}-4\\-12\\-1\end{pmatrix}\times\begin{pmatrix}-8\\-6\\1\end{pmatrix}=\begin{pmatrix}-18\\12\\-72\end{pmatrix}$$

b $\mathbf{p}=-\mathbf{q}$

24

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\left(\begin{pmatrix}3\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}\right)\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-7\\2\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-7\\3\\14\end{pmatrix}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{pmatrix}3\\1\\2\end{pmatrix}\times\left(\begin{pmatrix}1\\1\\3\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}\right)=\begin{pmatrix}3\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\5\\-2\end{pmatrix}=\begin{pmatrix}-12\\8\\14\end{pmatrix}$$

The two vector results are not the same.

25 **a**

$$\overrightarrow{AB}=\begin{pmatrix}2\\-7\\-2\end{pmatrix}=\overrightarrow{DC}$$
$$\overrightarrow{BC}=\begin{pmatrix}-2\\4\\7\end{pmatrix}=\overrightarrow{AD}$$

The quadrilateral has two pairs of parallel (and equal length) sides, so is a parallelogram

b

A parallelogram with side vectors **u** and **v** has area $|\mathbf{u} \times \mathbf{v}|$

$$\text{Area } ABCD = \left|\begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}\right| = \left|\begin{pmatrix} -41 \\ -10 \\ -6 \end{pmatrix}\right| = \sqrt{1817} \approx 42.6$$

26

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

A triangle with side vectors **u** and **v** has area $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$

$$\text{Area} = \frac{1}{2}\left|\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix} -1 \\ -6 \\ 3 \end{pmatrix}\right| = \frac{1}{2}\sqrt{46} \approx 3.39$$

27 **a**

$$\overrightarrow{BC} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix} = \overrightarrow{AD} \text{ so } D \text{ has coordinates } (11, -2, 0)$$

b

A parallelogram with side vectors **u** and **v** has area $|\mathbf{u} \times \mathbf{v}|$

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} = \overrightarrow{DC}$$

$$\text{Area } ABCD = \left|\begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}\right| = \left|\begin{pmatrix} -9 \\ -16 \\ -12 \end{pmatrix}\right| = \sqrt{481} \approx 21.9$$

28

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \cos\theta$

$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

where θ is the angle between vectors **a** and **b**

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 &= (|\mathbf{a}||\mathbf{b}|)^2(\cos^2\theta + \sin^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \end{aligned}$$

29

$\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$ so $\mathbf{r} = -\mathbf{p} - \mathbf{q}$

$$\begin{aligned} \mathbf{q} \times \mathbf{r} &= \mathbf{q} \times (-\mathbf{p} - \mathbf{q}) \\ &= \big(\mathbf{q} \times (-\mathbf{p})\big) - (\mathbf{q} \times \mathbf{q}) (\text{distributing}) \\ &= \mathbf{p} \times \mathbf{q} - \mathbf{0} (\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \mathbf{u} \times \mathbf{u} = \mathbf{0}) \\ &= \mathbf{p} \times \mathbf{q} \end{aligned}$$

$$\begin{aligned} \mathbf{r} \times \mathbf{p} &= (-\mathbf{p} - \mathbf{q}) \times \mathbf{p} \\ &= (-\mathbf{p} \times \mathbf{p}) + (-\mathbf{q} \times \mathbf{p}) (\text{distributing}) \\ &= \mathbf{p} \times \mathbf{q} - \mathbf{0} (\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \mathbf{u} \times \mathbf{u} = \mathbf{0}) \\ &= \mathbf{p} \times \mathbf{q} \end{aligned}$$

30 **a**
$C(5,4,0), F(5,0,2), G(5,4,2), H(0,4,2)$

b
$\overrightarrow{BE} = \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{BG} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$

$$\begin{aligned} \text{Area } BEG &= \frac{1}{2}\left|\overrightarrow{BE} \times \overrightarrow{BG}\right| \\ &= \frac{1}{2}\left|\begin{pmatrix} -8 \\ 10 \\ -20 \end{pmatrix}\right| \\ &= \frac{1}{2}\sqrt{564} \approx 11.9 \end{aligned}$$

Exercise 8G

13
Solving

$$\begin{cases} 2 + 4\lambda - \mu = -4 & (1) \\ 1 + \lambda + 4\mu = 8 & (2) \\ 1 + 2\lambda + 7\mu = 13 & (3) \end{cases}$$

$(3) - 2(2): -\mu - 1 = -3 \Rightarrow \mu = 2$
$(2): \lambda = 7 - 4\mu = -1$
Substituting into equation (1):
$2 + 4\lambda - \mu = 2 - 4 - 2 = -4$
The point $(-4, 8, 13)$ lies on the plane.

14 **a**

$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Π has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

b
Solving

$$\begin{cases} 3 - \mu = 4 & (1) \\ -1 + 2\lambda + \mu = 4 & (2) \\ 2 + \lambda + \mu = 0 & (3) \end{cases}$$

$(1): \mu = -1$
$(3): \lambda = -\mu - 2 = -1$
Substituting into equation (2):
$-1 + 2\lambda + \mu = -1 - 2 - 1 \neq 4$
The point $(4, 4, 0)$ does not lie on the plane.

15 **a**

The plane is described by $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$$\mathbf{a}\cdot\mathbf{n}=\begin{pmatrix}1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\7\end{pmatrix}=39$$

The plane has Cartesian equation $4x-y+7z=39$

b

$\mathbf{b}\cdot\mathbf{n}=2\neq 39$ so B does not lie in the plane.

16 **a** $\mathbf{n}=\begin{pmatrix}5\\1\\-4\end{pmatrix}$

b

Plane Π has scalar product form $\mathbf{r}\cdot\begin{pmatrix}5\\1\\-4\end{pmatrix}=20$

Require $\begin{pmatrix}2\\c\\1\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-4\end{pmatrix}=20$

$6+c=20$

$c=14$

17 **a**

From the coefficients of the Cartesian equation, the normal vector is $\mathbf{n}=\begin{pmatrix}1\\5\\-8\end{pmatrix}$

$|\mathbf{n}|=\sqrt{90}=3\sqrt{10}$

A unit normal vector is $\frac{\sqrt{10}}{30}\begin{pmatrix}1\\5\\-8\end{pmatrix}$

b

Π has scalar product form $\mathbf{r}\cdot\begin{pmatrix}1\\5\\-8\end{pmatrix}=37$

Require $\begin{pmatrix}p\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\5\\-8\end{pmatrix}=37\Rightarrow p+7=37\Rightarrow p=30$

Require $\begin{pmatrix}48\\q\\2\end{pmatrix}\cdot\begin{pmatrix}1\\5\\-8\end{pmatrix}=37\Rightarrow 5q+32=37\Rightarrow q=1$

c

The line connecting $(p,3,1)$ and $(48,q,2)$ has direction vector $\begin{pmatrix}18\\-2\\1\end{pmatrix}$ and passes through $(30,3,1)$

It therefore has vector equation $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}30\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}-18\\2\\-1\end{pmatrix}$

Solving for λ:

$$\lambda=\frac{x-30}{-18}=\frac{y-3}{2}=\frac{z-1}{-1}$$

The Cartesian equation of the line is $\frac{x-30}{-18}=\frac{y-3}{2}=\frac{z-1}{-1}$

18 **a**

Solving for $\mathbf{r_1} = \mathbf{r_2}$

$$\begin{pmatrix}7\\-3\\2\end{pmatrix} + t\begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}1\\1\\26\end{pmatrix} + s\begin{pmatrix}0\\1\\3\end{pmatrix}$$

$$\begin{cases}7 - t = 1 & (1)\\ -3 = 1 + s & (2)\\ 2 + 2t = 26 + 3s & (3)\end{cases}$$

$(1): t = 6$

$(2): s = -4$

Substitute into (3):

$2 + 12 = 14 = 26 - 12$

Since the three equations are consistent for $t = 6, s = -4$ the two lines do intersect, at $(1, -3, 14)$

b

$$\begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\3\\-1\end{pmatrix}$$

c

$$\mathbf{r} \cdot \begin{pmatrix}-2\\3\\-1\end{pmatrix} = \begin{pmatrix}7\\-3\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\3\\-1\end{pmatrix}$$

$-2x + 3y - z = -25$

$2x - 3y + z = 25$

19 **a**

The first line has $x = 1 + 3\lambda, y = -1 + 4\lambda, z = 3 - 3\lambda$

Substituting these into the second line equation:

$$\frac{13 + 3\lambda}{2} = 4\lambda - 1 = 20 - 3\lambda$$

This has consistent solution $\lambda = 3$, so the two lines intersect at $(10, 11, -6)$

b

The two lines have direction vectors $\mathbf{d_1} = \begin{pmatrix}3\\4\\-3\end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix}2\\1\\1\end{pmatrix}$

$\mathbf{d_1} \times \mathbf{d_2} = \begin{pmatrix}7\\-9\\-5\end{pmatrix}$ is perpendicular to both lines.

c

From part **b**, the vector $\mathbf{n} = \begin{pmatrix}7\\-9\\-5\end{pmatrix}$ must be normal to the plane containing the two lines.

The plane has scalar product equation $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \cdot \mathbf{n}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}7\\-9\\-5\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}7\\-9\\-5\end{pmatrix}$$

$7x - 9y - 5z = 1$

20

The line l is $\mathbf{r} = \mathbf{a} + t\mathbf{d_1}$ for $A\ (9, -3, 7)$ and direction $\mathbf{d_1} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$

A second direction vector in the plane is $\mathbf{d_2} = \overrightarrow{AP} = \begin{pmatrix} 2 \\ 15 \\ 6 \end{pmatrix}$

A vector equation of the plane is $\mathbf{r} = \mathbf{a} + t\mathbf{d_1} + s\mathbf{d_2}$

$$\mathbf{r} = \begin{pmatrix} 9 \\ -3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 15 \\ 6 \end{pmatrix}$$

21 **a**

The line l passes through the origin, so the plane containing l and P is parallel to direction vector $\mathbf{d_2} = \overrightarrow{OP} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ as well as the line direction $\mathbf{d_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$2\mathbf{d_1} - \mathbf{d_2} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ so the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is parallel to the plane.

b

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

c

The plane has scalar product equation $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \mathbf{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$x - 2z = 0$

22

The plane $5x + y - 2z = 15$ has normal $\mathbf{n} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

and so can be expressed as $\mathbf{r} \cdot \mathbf{n} = 15$.

The line $\frac{x-4}{1} = \frac{y+1}{1}$ $= \frac{z-2}{3}$ has direction vector $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and passes through point $(4, -1, 2)$

The line can be expressed by vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Substituting the general position of a point on the line into the vector equation:

$$\left(\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = (20 - 1 - 4) + \lambda(5 + 1 - 6) = 15 + 0\lambda$$

That is, every point on the line satisfies the equation of the plane, so the line lies fully within the plane.

23

The line's direction vector $\mathbf{d_1} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ is one of the stated directions of the plane, so the line is parallel to the plane.

The point on the line with position vector $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$ lies in the plane, with $\lambda = 0, \mu = -1$

Since a point on the line lies in the plane and the line is parallel to the plane, the line must lie entirely within the plane.

24

$\overrightarrow{AB} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -15 \\ 6 \\ -6 \end{pmatrix} = -3\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

let $\mathbf{d_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and let $\mathbf{d_2} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

Plane ABC can be expressed by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1} + \mu\mathbf{d_2}$

$\mathbf{r} = \begin{pmatrix} 11 \\ 0 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

If D lies in the plane ABC then $\begin{pmatrix} 3 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$ for some λ and μ

$$\begin{cases} 11 + \lambda + 5\mu = 3 & (1) \\ 2\lambda - 2\mu = 8 & (2) \\ 6 + 3\lambda + 2\mu = 8 & (3) \end{cases}$$

$(2) + (3): 6 + 5\lambda = 16 \Rightarrow \lambda = 2$

$(2): \mu = \lambda - 4 = -2$

Substituting into (1):

$11 + 2 - 10 = 3$

The system of equations is so D must lie in the plane ABC.

There are many possible approaches to this question. A reasonable alternative would be to find the scalar product equation of the plane and show that D satisfies the equation:

$\mathbf{d_1} \times \mathbf{d_2} = \begin{pmatrix} 10 \\ 13 \\ -12 \end{pmatrix}$ so the equation of the plane ABC can be expressed as

$\mathbf{r} \cdot \begin{pmatrix} 10 \\ 13 \\ -12 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 13 \\ -12 \end{pmatrix}$

$10x + 13y - 12z = 38$

Substituting the coordinates of D: $x = 3, y = 8, z = 8$

$30 + 104 - 96 = 38$

D lies in the plane ABC

Exercise 8H

19 **a** $\mathbf{n_1} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

b

Acute angle θ between the planes is the same as the angle between the normals.

$\mathbf{n_2} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

$\cos\theta = \frac{|\mathbf{n_1}\cdot\mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{13}{\sqrt{11}\sqrt{51}}$

$\theta \approx 57°$

20 **a** $\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

c

The plane has equation $\mathbf{r}\cdot\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 11$

Substituting the line equation into the plane equation to find the value of λ at the intersection:

$\left(\begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\right)\cdot\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 11$

$-16 + 9\lambda = 11$

$\lambda = 3$ at the intersection point.

N has coordinates $(3, 3, 1)$

d $\overrightarrow{PN} = 3\mathbf{n}$ so $PN = 3|\mathbf{n}| = 9$

21 **a**

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 8 \\ -16 \\ 24 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

b

The normal of the plane is $\mathbf{n_1} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

The plane has equation $\mathbf{r}\cdot\mathbf{n_1} = \mathbf{a}\cdot\mathbf{n_1}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$x - 2y + 3z = 9$

c

Acute angle θ between the planes is the same as the angle between the normals.

$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{5}{\sqrt{14}\sqrt{14}}$

$\theta \approx 69.1°$

22 **a**

Line L has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Plane Π has equation $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 20$

Substituting the line equation into the plane equation to find the value of λ at the intersection:

$\left(\begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right) \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 20$

$15 + \lambda = 20$

$\lambda = 5$ at the intersection point.

M has coordinates $(9, 6, 7)$

b

If the angle between L and Π is θ then the angle between L and the plane normal is $90° - \theta$

$\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

$\cos(90° - \theta) = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \frac{1}{\sqrt{6}\sqrt{10}}$

$90° - \theta = 82.58°$

$\theta = 7.42°$

c

In the triangle AMN, $A\hat{N}M = 90°, A\hat{M}N = 7.42°, AM = 5|\mathbf{d}| = 5\sqrt{6}$

$MN = 5\sqrt{6}\cos 7.42° = 12.1$

23 **a**

If the angle between l and Π is θ then the angle between L and the plane normal is $90° - \theta$

Line direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and plane normal is $\mathbf{n} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

$\cos(90° - \theta) = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \frac{11}{\sqrt{10}\sqrt{42}}$

$90° - \theta = 57.5°$

$\theta = 32.5°$

b

Line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$, plane Π has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = 16$

Substituting the line equation into the plane equation to find the value of λ at the intersection:

$$\left(\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = 16$$

$-6 - 11\lambda = 16$

$\lambda = -2$ at the intersection point.

The intersection point has coordinates $(1, 1, 5)$

c

If the intersection point is M and the base of the perpendicular to the plane from A is N:

In the triangle AMN, $A\hat{N}M = 90°, A\hat{M}N = 32.5°, AM = 2|\mathbf{d}| = 2\sqrt{10}$

$AN = 2\sqrt{10} \sin 32.5° = 3.39$

24 **a**

The normal to the plane is $\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$, which must be the direction of the line.

The line has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

b

The plane has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = -24$

Substituting the line equation into the plane equation to find the value of λ at the intersection:

$$\left(\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = -24$$

$11 + 35\lambda = -24$

$\lambda = -1$ at the intersection point.

The intersection point has coordinates $(3, 4, -3)$

c

If the intersection point is B then part **b** shows that $\overrightarrow{AB} = -1 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ so if the reflection of A in the plane Π is C then $\overrightarrow{AC} = -2 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

The coordinates of C are $(2, 7, -8)$

25 **a**

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

b

The plane Π_1 has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$2x-y+z=0$

c

Substituting the coordinates of A into the plane equation:

If $x=3, y=4, z=-2$ then $3x+y-z=9+4+2=15$

Therefore $(3,4,-2)$ does lie in the plane $3x+y-z=15$

d

The line of intersection must pass through a point of intersection of the planes A in a direction perpendicular to both normals.

From part **a**, this direction vector is $\begin{pmatrix}0\\1\\1\end{pmatrix}$

The line of intersection of the planes has equation $\mathbf{r}=\begin{pmatrix}3\\4\\-2\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$

e

Substituting the line equation into the equation of Π_3 to find the value of λ at the intersection:

$$\left(\begin{pmatrix}3\\4\\-2\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}=12$$

$6+3\lambda=12$

$\lambda=2$ at the intersection point.

The intersection point has coordinates $(3,6,0)$

f

Acute angle θ between the planes is the same as the angle between the normals.

$\mathbf{n_1}=\begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{n_3}=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\cos\theta=\dfrac{|\mathbf{n_1}\cdot\mathbf{n_3}|}{|\mathbf{n_1}||\mathbf{n_3}|}=\dfrac{5}{\sqrt{6}\sqrt{9}}$

$\theta\approx 47.1°$

26 **a**

$$\begin{cases}3x-5y+z=7 & (1)\\ x+3y-4z=22 & (2)\\ 7x-21y+11z=a & (3)\end{cases}$$

Eliminating x:

$$\begin{cases}(1)-3(2): -14y+13z=-59 & (4)\\ (3)-7(2): -42y+39z=a-154 & (5)\end{cases}$$

$(5)-3(4): 0=a+23$

$a=-23$

b

When $a=-23$, the equations are linked such that $(3)-7(2)=3\big((1)-3(2)\big)$

That is, the system is consistent (has at least one solution), the planes are not parallel but there are infinitely many solutions to the three planes simultaneously.

The three planes intersect along a line.

The above argument is sufficient, but you could alternatively just find the line of intersection:

The normal vectors of the planes are $\mathbf{n_1} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$, $\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ and $\mathbf{n_3} = \begin{pmatrix} 7 \\ -21 \\ 11 \end{pmatrix}$

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 17 \\ 13 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} 17 \\ 13 \\ 14 \end{pmatrix} \cdot \mathbf{n_3} = 119 - 273 + 154 = 0$$

The direction vector $\begin{pmatrix} 17 \\ 13 \\ 14 \end{pmatrix}$ is perpendicular to all three normals so is parallel to all three planes.

Since the planes are consistent (have at least one intersection point) and have distinct normal directions, and are all parallel to the same direction vector, they must intersect in a line.

The line has equation $\mathbf{r} = \begin{pmatrix} 13 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 17 \\ 13 \\ 14 \end{pmatrix}$

27 **a**

$$\begin{cases} x - y - 2z = 2 & (1) \\ 2x - 2y + z = 0 & (2) \\ 3x - 3y + 4z = a & (3) \end{cases}$$

Eliminating x:

$$\begin{cases} (2) - 2(1): 5z = -4 & (4) \\ (3) - 3(1): 10z = a - 6 & (5) \end{cases}$$

If $a = 1$ then the system is inconsistent.

The three planes have distinct normal directions, so the three planes enclose a triangular prism.

b

If $a = 2$ then the system is consistent and has infinitely many solutions, so the planes intersect in a line.

The normal vectors of the planes are $\mathbf{n_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$, $\mathbf{n_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{n_3} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The direction vector of the line of intersection is $\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

For a point on the intersection line:

$(4) \Rightarrow z = -0.8$

$(1) \Rightarrow x - y = 0.4$

A point on the line is $(0.4, 0, -0.8)$

The line has equation $\mathbf{r} = \begin{pmatrix} 0.4 \\ 0 \\ -0.8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

28 **a**

The line of intersection passes through a point of intersection A in a direction $\mathbf{d}$ which is perpendicular to both plane normals.

$\mathbf{n_1} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \mathbf{d}$

For a point A, fix $x = 0$:

$\begin{cases} -3y + z = 7 & (1) \\ -2y + z = 10 & (2) \end{cases}$

$(2) - (1): y = 3$

$y = 3, z = 16$

Line has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

b

The normal to plane Π_3 is $\mathbf{n_3} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$

$\mathbf{d} \cdot \mathbf{n_3} = -5 - 7 + 12 = 0$

The line is perpendicular to the normal of Π_3, so is parallel to the plane itself.

c

Substituting $x = 0, y = 3, z = 16$ into the equation for Π_3:

$5x - 7y + 3z = 0 - 21 + 48 = 27 \neq 16$ so point A does not line in Π_3

The planes do not have a common line of intersection, but each plane is parallel to direction $\mathbf{d}$.

The planes form a triangular prism.

29 **a**

Substituting $x = -3, y = 4, z = c$ into the two plane equations:

$x - 4z + 7 = -3 - 4c + 7 \Rightarrow c = 1$

$4x + 5y - z = -12 + 20 - c = 7 \Rightarrow c = 1$

Hence $(-3, 4, 1)$ lies in both planes.

b

$\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -15 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

c

The direction of the line of intersection $\mathbf{d} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$, and from part **a**, the line passes through $(-3, 4, 1)$.

The line has Cartesian equation

$\frac{x + 3}{4} = \frac{y - 4}{-3} = z - 1$

30 **a**

$\overrightarrow{AB} = \begin{pmatrix} -7 \\ -4 \\ -1 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -2 \\ -12 \\ 62 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}$

b

Area $ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{1^2 + 6^2 + (-31)^2} = \sqrt{998} \approx 31.6$

c

The normal to the plane is $\mathbf{n} = \begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}$

The scalar product form of the plane equation is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}$

$x + 6y - 31z = -86$

d

Substitute $x = -5, y = 2, z = 11$ into the plane equation in part **c**:

$x + 6y - 31z = -5 + 12 - 341 \neq -86$ so D is not a point in the plane.

e

Line l has vector equation $\mathbf{r} = \begin{pmatrix} -5 \\ 2 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}$

Substituting the line equation into the scalar product plane equation to find the value of λ at the intersection:

$\left(\begin{pmatrix} -5 \\ 2 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix} = -86$

$-334 + 998\lambda = -86$

$\lambda = \frac{248}{998} \approx 0.248$ at the intersection point

Intersection point has approximate coordinates $(-4.75, 3.49, 3.30)$

f

$$\begin{aligned}\text{Volume } ABCD &= \frac{1}{3}\text{base area} \times \text{height} \\ &= \frac{1}{3}\sqrt{998} \times 0.248|\mathbf{n}| \\ &= \frac{1}{3}\sqrt{998} \times \frac{248}{998} \times \sqrt{998} \\ &= \frac{248}{3}\end{aligned}$$

You can alternatively use the ‘triple product’ formula for the volume of a tetrahedron:

$V = \frac{1}{6}|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$

where **u**,· **v** and **w** are any three edge vectors.

We already have

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -2 \\ -12 \\ 62 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix} -8 \\ -9 \\ 6 \end{pmatrix}$$

$$\text{Volume} = \frac{1}{6}\left|2\begin{pmatrix} 1 \\ 6 \\ -31 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ -9 \\ 6 \end{pmatrix}\right|$$
$$= \frac{248}{3}$$

See if you can prove the validity of the formula!

31 **a**

Using the coefficients of the Cartesian form, the normals are

$$\mathbf{n_1} = \begin{pmatrix} 6 \\ -9 \\ 15 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$
$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 6 \\ -10 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

So the two planes have a common normal direction $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and are therefore parallel.

b

Substituting $x = 2, y = 0, z = k$ into the equation for Π_1:

$12 - 0 + 15k = 20$

$k = \frac{8}{15}$

c

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 8/15 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

d

This line intersects Π_1 at A, represented by $\lambda = 0$.

Substituting the line equation into the scalar product plane equation of Π_2 to find the value of λ at the intersection:

$$\left(\begin{pmatrix} 2 \\ 0 \\ 8/15 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right) \cdot \begin{pmatrix} -4 \\ 6 \\ -10 \end{pmatrix} = 3$$

$-8 - \frac{16}{3} - 76\lambda = 3$

$76\lambda = \frac{49}{3}$ at the intersection point: $\lambda = \frac{49}{228}$

The distance between the planes is $\lambda\left|\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right| = \frac{49}{6\sqrt{38}} \approx 1.32$

32 **a**

The normal to Π_1 is $\mathbf{n_1} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$

The normal to Π_2 is $\mathbf{n_2} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$

$\mathbf{n_1} \cdot \mathbf{n_2} = 5 - 3 - 2 = 0$

Since the two normals are perpendicular, it follows that the two planes are perpendicular.

b

$$\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \\ -16 \end{pmatrix}$$

c

Substituting $x = 1, y = -1, z = 5$ into eacb plane equation:

$5x + y + z = 5 - 1 + 5 = 9 \neq 21$ so P does not lie in Π_1

$x - 3y - 2z = 1 + 3 - 10 = -6 \neq 3$ so P does not lie in Π_2

d

The line direction must be perpendicular to both normals if the line is to be parallel to both planes.

Direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 11 \\ -16 \end{pmatrix}$ from part **b**

The line passes through $(1, -1, -5)$

The Cartesian equation of the line is $x - 1 = \frac{y + 1}{11} = \frac{z + 5}{-16}$

33

$$\begin{cases} 3x - y + 5z = 2 & (1) \\ 2x + 4y + z = 1 & (2) \\ x + y + kz = c & (3) \end{cases}$$

Eliminating x:

$$\begin{cases} (1) - 3(3): -4y + (5 - 3k)z = 2 - 3c & (4) \\ (2) - 2(3): 2y + (1 - 2k)z = 1 - 2c & (5) \end{cases}$$

$(4) + 2(5): (7 - 7k)z = 4 - 7c \quad (6)$

a

If the planes intersect at a unique point, there must be a unique solution to equation (6):

$k \neq 1$

b

If the planes intersect along a line then the system must be consistent but with no unique solution:

$k = 1, c = \frac{4}{7}$

c

If there should be no intersection, the system must be inconsistent:

$k = 1, c \neq \frac{4}{7}$

Mixed Practice

1 **a** $\overrightarrow{MD} = \frac{1}{2}\mathbf{b} - \mathbf{a}$

b $\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN}$

$$= \mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\overrightarrow{MD}$$

$$= \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$$

c $\overrightarrow{AP} = \mathbf{a} + \frac{3}{2}\mathbf{b} = 2\overrightarrow{AN}$

This shows that A, P and N are collinear, with P the midpoint of AN.

2 **a**

$$\mathbf{a} = \begin{pmatrix}3\\-1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\5\\-2\end{pmatrix}, \mathbf{c} = \begin{pmatrix}3\\1\\-1\end{pmatrix}, \mathbf{d} = \begin{pmatrix}4\\-1\\p\end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}-3\\7\\16\end{pmatrix}$$

$$\begin{pmatrix}-3\\7\\16\end{pmatrix} \cdot \mathbf{c} = -18$$

b

Require $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = 0$

$$\begin{pmatrix}-3\\7\\16\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\p\end{pmatrix} = -19 + 16p$$

$$p = \frac{19}{16}$$

3

Require $\begin{pmatrix}3\sin x\\8\\1\end{pmatrix} \cdot \begin{pmatrix}4\cos x\\1\\-2\end{pmatrix} = 0$

$12\sin x\cos x + 6 = 0$

$2\sin x\cos x = -1$

$\sin 2x = -1$

$2x = \frac{3\pi}{2} + 2n\pi$

The only solution for $0 < x < \pi$ is $x = \frac{3\pi}{4}$

4 **a** $\mathbf{a} \times \mathbf{c} = \begin{pmatrix}0\\2\\-5\end{pmatrix} \times \begin{pmatrix}1\\0\\3\end{pmatrix} = \begin{pmatrix}6\\-5\\-2\end{pmatrix}$

b $\mathbf{b} = \mathbf{a} + \mathbf{c} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ so B has coordinates $(1, 2, -2)$

c Area $OABC = |\mathbf{a} \times \mathbf{c}| = \left|\begin{pmatrix}6\\-5\\-2\end{pmatrix}\right| = \sqrt{65}$

5 **a** $|\mathbf{v}| = \sqrt{116^2 + 52^2 + 12^2} = \sqrt{16304} \approx 128 \text{ m s}^{-1}$

b The z coordinate will equal 1000 at time $t = \frac{1000}{12} \approx 83.3$ s

6 **a** $\overrightarrow{AD} = \begin{pmatrix} 2 \\ 0 \\ k-7 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}$

$\overrightarrow{AD} \cdot \overrightarrow{AB} = 0 = -4 - 4(k-7)$

$k - 7 = -1$

$k = 6$

c $\overrightarrow{BC} = 2\overrightarrow{AD} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ so C has coordinates $(3, 6, 1)$

d $\overrightarrow{DA} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{DC} = \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix}$

$\cos(A\hat{D}C) = \frac{\overrightarrow{DA} \cdot \overrightarrow{DC}}{|\overrightarrow{DA}||\overrightarrow{DC}|} = \frac{-5}{\sqrt{5}\sqrt{50}} = -\frac{\sqrt{10}}{10}$

7 **a**

Direction $\mathbf{d_1} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

A vector equation of the line is $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

b $\mathbf{d_2} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

Angle θ between the lines is such that

$\cos\theta = \frac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|} = -\frac{1}{\sqrt{27}\sqrt{51}}$

$\theta = 88.5°$

c

Require $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ k \end{pmatrix}$

$\lambda = 3$ so $k = 24$

d

$AC = \sqrt{6^2 + 6^2 + 22^2} = \sqrt{556} \approx 23.6$

8 **a**

$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -3 \end{pmatrix}$

b

Point of intersection occurs where $\begin{pmatrix}5\\1\\2\end{pmatrix}+t\begin{pmatrix}-1\\1\\3\end{pmatrix}=\begin{pmatrix}5\\4\\9\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix}$

$$\begin{cases}5-t=5+2s & (1)\\ 1+t=4+s & (2)\\ 2+3t=9+s & (3)\end{cases}$$

$(1)+(2)$: $6=9+3s\Rightarrow s=-1$

(2): $t=3+s=2$

Substituting into (3):

$2+3t=8=9+s$

The system is consistent for $s=-1, t=2$ so there is an intersection point, at $(3,3,8)$

c

From part **a**, $\mathbf{n}=\begin{pmatrix}2\\-7\\3\end{pmatrix}$ is perpendicular to the plane containing the two lines.

d

The plane is given by $\mathbf{r}\cdot\mathbf{n}=\begin{pmatrix}5\\1\\2\end{pmatrix}\cdot\mathbf{n}$

The Cartesian form is $2x-7y+3z=9$

9 **a**

The plane is given by $\mathbf{r}\cdot\mathbf{n}=\begin{pmatrix}3\\-1\\2\end{pmatrix}\cdot\mathbf{n}$

$3x+y-z=6$

b

Substituting $x=a, y=2a, z=a-1$ into the equation in part **a**:

$3a+2a+1-a=6$

$4a=5$

$a=1.25$

10 **ai** $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AC}=\frac{1}{2}(\mathbf{c}-\mathbf{a})$

aii $\overrightarrow{BM}=\overrightarrow{BA}+\overrightarrow{AM}=(\mathbf{a}-\mathbf{b})+\frac{1}{2}(\mathbf{c}-\mathbf{a})=\frac{1}{2}(\mathbf{a}-2\mathbf{b}+\mathbf{c})$

bi $\overrightarrow{RA}=\frac{1}{3}\overrightarrow{BA}=\frac{1}{3}(\mathbf{a}-\mathbf{b})$

bii $\overrightarrow{AS}=\frac{2}{3}\overrightarrow{AC}=\frac{2}{3}(\mathbf{c}-\mathbf{a})$

$\overrightarrow{RT}=\frac{2}{3}\overrightarrow{RS}=\frac{2}{3}\left(\overrightarrow{RA}+\overrightarrow{AS}\right)=\frac{2}{9}\left(\mathbf{a}-\mathbf{b}+2(\mathbf{c}-\mathbf{a})\right)=\frac{2}{9}(-\mathbf{a}-\mathbf{b}+2\mathbf{c})$

c
$$\begin{aligned}\overrightarrow{BT}&=\overrightarrow{BR}+\overrightarrow{RT}\\&=\frac{2}{3}\overrightarrow{BA}+\overrightarrow{RT}\\&=\frac{2}{3}(\mathbf{a}-\mathbf{b})+\frac{2}{9}(-\mathbf{a}-\mathbf{b}+2\mathbf{c})\\&=\frac{4}{9}(\mathbf{a}-2\mathbf{b}+\mathbf{c})\\&=\frac{8}{9}\overrightarrow{BM}\end{aligned}$$

Hence T is on the line BM.

11

$$\left|\begin{pmatrix}2\\-1\\1\end{pmatrix}\right| = \sqrt{6}$$

The unit vector is therefore $\frac{\sqrt{6}}{6}\begin{pmatrix}2\\-1\\1\end{pmatrix}$

12

$\mathbf{a} \times \mathbf{b} = k\mathbf{c}$

$\begin{pmatrix}-5\\-1-3p\\15\end{pmatrix} = k\begin{pmatrix}1\\4\\-3\end{pmatrix}$

$k = -5$

$-1 - 3p = -20$

$p = \frac{19}{3}$

13 **a**

$\overrightarrow{AB} = \begin{pmatrix}k-2\\0\\2k-1\end{pmatrix} = \overrightarrow{DC}$

$\overrightarrow{AD} = \begin{pmatrix}4\\2k\\2\end{pmatrix} = \overrightarrow{CB}$

None of these vectors can be the zero vector so, by definition, $ABCD$ is a parallelogram.

b

$k = 1 \Rightarrow \overrightarrow{AB} = \begin{pmatrix}-1\\0\\1\end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix}4\\2\\2\end{pmatrix}$

Angle A is such that $\cos B\hat{A}D = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|} = \frac{-2}{\sqrt{2}\sqrt{24}}$

$B\hat{A}D = 107° = B\hat{C}D$

$A\hat{B}C = A\hat{D}C = 73.2°$

c

If $ABCD$ is a rectangle then $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$

$4(k-2) + 2(2k-1) = 0$

$8k - 10 = 0$

$k = \frac{5}{4}$

14

Properties of vector product: $\mathbf{u} \times \mathbf{u} = \mathbf{0}, \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

$$\begin{aligned}(2\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + 3\mathbf{b}) &= \mathbf{2a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} - \mathbf{3b} \times \mathbf{b}\\ &= \mathbf{0} + 6\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} - \mathbf{0}\\ &= 7\mathbf{a} \times \mathbf{b}\end{aligned}$$

15 Area of a triangle with side vectors **a** and **b** is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

Two side vectors are $\begin{pmatrix}-3\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\4\end{pmatrix}$

$$\text{Area} = \frac{1}{2}\left|\begin{pmatrix}-3\\1\\1\end{pmatrix} \times \begin{pmatrix}-2\\0\\4\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}4\\10\\2\end{pmatrix}\right| = \frac{1}{2}\sqrt{120} = \sqrt{30}$$

16

$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2q \\ 1 \\ q \end{pmatrix}$

Require $p\mathbf{a} + \mathbf{b} = k\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$\begin{cases} p + 2q = k & (1) \\ -p + 1 = k & (2) \\ 3p + q = 2k & (3) \end{cases}$$

Eliminating k:

$$\begin{cases} (1) - (2): & 2p + 2q - 1 = 0 & (4) \\ (3) - 2(2): & 5p + q - 2 = 0 & (5) \end{cases}$$

$2(5) - (4): 8p - 3 = 0 \Rightarrow p = \frac{3}{8}$

$(5): q = 2 - 5p = \frac{1}{8}$

17 **a**

$|\mathbf{a}| = |\mathbf{b}| = 1$ so $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1$ and $\mathbf{a} \cdot \mathbf{b} = \cos\alpha$

$$\begin{aligned} |\mathbf{a} - \mathbf{b}| &= \sqrt{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b}} \\ &= \sqrt{2 - 2\cos\alpha} \end{aligned}$$

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{b}} \\ &= \sqrt{2 + 2\cos\alpha} \end{aligned}$$

b

$|\mathbf{a} + \mathbf{b}| = 4|\mathbf{a} - \mathbf{b}|$

$|\mathbf{a} + \mathbf{b}|^2 = 16|\mathbf{a} - \mathbf{b}|^2$

$2 + 2\cos\alpha = 32 - 32\cos\alpha$

$34\cos\alpha = 30$

$\alpha = \cos^{-1}\left(\frac{15}{17}\right) \approx 28.1°$

18 **a**

The direction vectors are not multiples of each other, so the lines are not parallel.

$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 4 + 2 - 6 = 0$ so the two lines directions are perpendicular.

b

Solving for an intersection:

$$\begin{cases} 2+4\lambda = -2+\mu & (1) \\ -1-\lambda = -2\mu & (2) \\ 5+2\lambda = 3-3\mu & (3) \end{cases}$$

$2(1)+(2): 3+7\lambda = -4 \Rightarrow \lambda = -1$

$(2): \mu = \frac{1}{2}(1+\lambda) = 0$

Substituting into $(3): 5+2\lambda = 3 = 3-3\mu$

The system is consistent so the two lines do intersect and are perpendicular.

19 **a**

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

bi

Solving for an intersection:

$$\begin{cases} 7-t = 1 & (1) \\ -3 = 1+s & (2) \\ 2+2t = 26+3s & (3) \end{cases}$$

$(1) \Rightarrow t = 6$

$(2): s = -4$

Substituting into $(3): 2+2t = 14 = 26+3s$

The system is consistent so the two lines do intersect

bii

Substituting $t = 6$ into the first line equation gives the intersection point $(1, -3, 14)$

c

From part **a**, the normal to the plane is $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

The plane has scalar product equation $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix} \cdot \mathbf{n}$

$2x - 3y + z = 25$

20 **a**

Rewriting the Cartesian form:

$$\frac{x-1.5}{1} = \frac{y-3}{-4} = \frac{z-0}{5}$$

The vector form of the equation is $\mathbf{r} = \begin{pmatrix} 1.5 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

b

$\mathbf{a} = \begin{pmatrix} 1.5 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$ and $\mathbf{a} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 0$

$\left(\begin{pmatrix} 1.5 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = -10.5 + 42\lambda = 0$

$\lambda = \frac{1}{4}$

A has coordinates $\left(\frac{7}{4}, 2, \frac{5}{4}\right)$

c

Shortest distance between l and O is $OA = \frac{1}{4}\sqrt{7^2 + 8^2 + 5^2} = \frac{1}{4}\sqrt{138} \approx 2.94$

21 **a**

l_1 has direction vector $\mathbf{d} = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$. If l_2 is parallel to l_1 through $(0, -1, 2)$ then l_2 has equation

$\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

b

Require $\overrightarrow{AB} \cdot \mathbf{d} = 0$

B has position vector $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

$\left(\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} \right) \cdot \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} = 2 + 34\mu = 0 \Rightarrow \mu = -\frac{1}{17}$

B has coordinates $\left(\frac{30}{17}, -\frac{14}{17}, -\frac{3}{17}\right)$

c $\overrightarrow{AB} = \frac{1}{17}\begin{pmatrix} 30 \\ 3 \\ -37 \end{pmatrix}$ so $AB = \frac{1}{17}\sqrt{30^2 + 3^2 + 37^2} \approx 47.7$

22 **a**

$\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

b

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

Angle θ between the direction $\overrightarrow{AB}$ and the direction vector $\mathbf{l}$ of the line is such that

$\cos\theta = \left| \frac{\overrightarrow{AB} \cdot \mathbf{l}}{|\overrightarrow{\mathrm{AB}}||\mathbf{l}|} \right| = \frac{3}{\sqrt{18}\sqrt{2}} = \frac{1}{2}$

c $AB = \sqrt{18} = 3\sqrt{2}$

d $C\hat{A}B = \cos^{-1}\left(\frac{1}{2}\right) = 60°, A\hat{C}B = 90°, AB = 3\sqrt{2}$

$$AC = 3\sqrt{2} \times \frac{1}{2} = \frac{3\sqrt{2}}{2}$$

23 **a**

Rewriting the Cartesian form of the second line:

$$\frac{x-1}{4} = \frac{y+2}{3} = \frac{z-0.5}{2}$$

The vector equation of the line is $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 0.5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

Solving for an intersection:

$$\begin{cases} 4 - 3\lambda = 1 + 4\mu & (1) \\ 1 + 3\lambda = -2 + 3\mu & (2) \\ 2 + \lambda = 0.5 + 2\mu & (3) \end{cases}$$

$(1) + (2) \Rightarrow 5 = 7\mu - 1 \Rightarrow \mu = \frac{6}{7}$

$(2): 3\lambda = -3 + 3\mu \Rightarrow \lambda = -\frac{1}{7}$

Substituting into (3): $2 + \lambda = \frac{13}{7} \neq 0.5 + 2\mu$

The system is inconsistent so the two lines do not intersect

b

$$\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -21 \end{pmatrix}$$

c

From part **b**, the normal to the plane will be $\mathbf{n} = \begin{pmatrix} 3 \\ 10 \\ -21 \end{pmatrix}$

If the plane passes through $(3, 0, 1)$ then it has scalar product equation $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \mathbf{n}$

$3x + 10y - 21z = -12$

24 **a**

Rewriting the Cartesian form:

$$\frac{x - \frac{1}{2}}{2} = \frac{y+2}{3} = \frac{z - \frac{4}{3}}{-2}$$

The line passes through $\left(\frac{1}{2}, -2, \frac{4}{3}\right)$ with direction $\mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

The vector equation of the line is $\mathbf{r} = \frac{1}{6}\begin{pmatrix} 3 \\ -12 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

b

If the line intersects the x-axis then $\mathbf{r} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ for some value a.

Solving for λ:

$$\begin{cases} \frac{1}{2} + 2\lambda = a & (1) \\ -2 + 3\lambda = 0 & (2) \\ \frac{4}{3} - 2\lambda = 0 & (3) \end{cases}$$

(2)and (3)both have solution $\lambda = \frac{2}{3}$

The line does intersect the x-axis.

The point of intersection is $\left(\frac{11}{6}, 0, 0\right)$

c

The angle between line and x-axis is θ where

$\cos\theta = \frac{\mathbf{d} \cdot \mathbf{i}}{|\mathbf{d}|} = \frac{2}{\sqrt{17}}$

$\theta = 61.0°$

25 **a**

L passes through $(3, 1, -4)$ with direction $\mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

The line has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

In Cartesian form, this is $\frac{x-3}{3} = \frac{y-1}{-1} = \frac{z+4}{-1}$

b

Substituting $x = 3 + 3\lambda, y = 1 - \lambda, z = -4 - \lambda$ into the plane equation:

$3(3 + 3\lambda) - (1 - \lambda) - (-4 - \lambda) = 1$

$11\lambda = -11$

$\lambda = -1$

The intersection point is $P(0, 2, -3)$

c

A is equivalent to $\lambda = 0$ on the line equation and the plane is intersected at P given by $\lambda = -1$

Then the image of A is at $\lambda = -2$, which has coordinates $(-3, 3, -2)$

d

Substituting $x = y = z = 1$ into the plane equation:

$3x - y - z = 3 - 1 - 1 = 1$

Therefore $B(1, 1, 1)$ lies in the plane.

e $BP = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$

26 **a**

$$\mathbf{r} = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Displacement at $t = 5$ is $5\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \\ 10 \end{pmatrix}$

b

Speed is $\left|\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right| = \sqrt{14} \approx 3.74 \text{ m s}^{-1}$

c

The given line has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$

Solving for an intersection:

$$\begin{cases} 3 - 3\lambda = 4 + 3t & (1) \\ -\lambda = 6 - t & (2) \\ 1 + 4\lambda = -7 + 2t & (3) \end{cases}$$

$(1) - 3(2) \Rightarrow 3 = -14 + 6t \Rightarrow t = \frac{17}{6}$

$(2): \lambda = t - 6 \Rightarrow \lambda = -\frac{19}{6}$

Substituting into (3): $1 + 4\lambda = -\frac{35}{3} \neq -7 + 2t$

The system is inconsistent so the particle's path does not cross that line.

27 **a**

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

b

$$\mathbf{r_2} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{r_2} - \mathbf{r_1} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$$

$$\begin{aligned} d^2 &= |\mathbf{r_2} - \mathbf{r_1}|^2 \\ &= (0 + 2t)^2 + (-5 + 6t)^2 + (7 - 2t)^2 \\ &= 4t^2 + 25 - 60t + 36t^2 + 49 - 28t + 4t^2 \\ &= 44t^2 - 88t + 74 \end{aligned}$$

c

Completing the square:

$$\begin{aligned} d^2 &= 44(t^2 - 2t) + 74 \\ &= 44(t - 1)^2 - 44 + 74 \\ &= 44(t - 1)^2 + 30 \end{aligned}$$

Hence $d^2 \geq 30$ at all times and the two aircraft cannot collide

d The minimum value of d is $\sqrt{30} \approx 5.48$ km

28

x-axis intercept: $y = z = 0$ so $3x = 30$. Point is $(10, 0, 0)$
y-axis intercept: $x = z = 0$ so $-y = 30$. Point is $(0, -30, 0)$
z-axis intercept: $x = y = 0$ so $5z = 30$. Point is $(0, 0, 6)$

Area of a triangle is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors of the two of the edge vectors.

Two of the side vectors are $\begin{pmatrix}-10\\-30\\0\end{pmatrix}$ and $\begin{pmatrix}-10\\0\\6\end{pmatrix}$

$$\text{Area} = \frac{1}{2}\left|\begin{pmatrix}-10\\-30\\0\end{pmatrix} \times \begin{pmatrix}-10\\0\\6\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}-180\\60\\-300\end{pmatrix}\right| = \frac{1}{2}\sqrt{12600} = 30\sqrt{35} \approx 177$$

29 **a**

$\mathbf{u} \times \mathbf{u} = \mathbf{0}, \mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ and $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ for any vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

For the unit vectors, $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$

$$(\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= \mathbf{i} \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) - \mathbf{j} \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + 4\mathbf{k} \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= 2\mathbf{i} \times \mathbf{i} - \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} - 2\mathbf{j} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} - \mathbf{j} \times \mathbf{k} + 8\mathbf{k} \times \mathbf{i} - 4\mathbf{k} \times \mathbf{j} + 4\mathbf{k} \times \mathbf{k}$$
$$= \mathbf{0} - \mathbf{k} - \mathbf{j} + 2\mathbf{k} + \mathbf{0} - \mathbf{i} + 8\mathbf{j} + 4\mathbf{i} + \mathbf{0}$$
$$= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$$

b

$$\begin{pmatrix}2\\2\\3\end{pmatrix} \cdot \mathbf{a} = \begin{pmatrix}2\\2\\3\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\1\end{pmatrix} = 4 - 2 + 3 = 5 \text{ so } (2, 2, 3) \text{ lies in } \mathbf{r} \cdot \mathbf{a} = 5$$
$$\begin{pmatrix}2\\2\\3\end{pmatrix} \cdot \mathbf{b} = \begin{pmatrix}2\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\4\end{pmatrix} = 2 - 2 + 12 = 12 \text{ so } (2, 2, 3) \text{ lies in } \mathbf{r} \cdot \mathbf{b} = 12$$

c

The line of intersection of planes has direction perpendicular to both plane normals.

From part **a**, the direction of the line is $\mathbf{d} = \begin{pmatrix}3\\7\\1\end{pmatrix}$

The line has vector equation $\mathbf{r} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\7\\1\end{pmatrix}$

In Cartesian form, this is $\frac{x-2}{3} = \frac{y-2}{7} = z - 3$

30 **a**

$\overrightarrow{AB}=\begin{pmatrix}1\\-1\\3\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}2\\0\\1\end{pmatrix}, \overrightarrow{AD}=\begin{pmatrix}-1\\5\\2\end{pmatrix}$

$\overrightarrow{AB}\cdot\overrightarrow{AD}=\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=-1-5+6=0$ so $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AD}$

$\overrightarrow{AC}\cdot\overrightarrow{AD}=\begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=-2+0-2=0$ so $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{AD}$

b

From part **a**, the normal to the plane containing A, B and C has normal vector $\mathbf{n}=\overrightarrow{AD}$
The plane has vector equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$\mathbf{r}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=-5$

c $AD=\sqrt{(-1)^2+5^2+2^2}=\sqrt{30}$

d $\text{Volume}=\frac{1}{3}(\text{base area})\times\text{height}$

$$=\frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right)\times\left|\overrightarrow{AD}\right|$$
$$=\frac{1}{3}\left(\frac{1}{2}\sqrt{30}\right)\times\sqrt{30}$$
$$=5$$

31

Substituting $x=0$, the point $(0,-20,-25)$ lies on the line.
If the line is the intersection of the planes then this point must lie in both planes.
Substituting into the second plane equation,
$4(0)+(-20)-(-25)=k$
$k=5$

32 **a**

The angle θ between the direction vectors $\mathbf{d_1}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$ and $\mathbf{d_2}=\begin{pmatrix}1\\5\\-3\end{pmatrix}$ satisfies

$\cos\theta=\left|\frac{\mathbf{d_1}\cdot\mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}\right|=\left|\frac{-11}{\sqrt{9}\sqrt{35}}\right|$
$\theta=51.7°$

b

Solving for the intersection:

$$\begin{cases}5+2\lambda=\mu & (1)\\ -3-2\lambda=7-6\mu & (2)\\ 1+\lambda=-5+4\mu & (3)\end{cases}$$

$(1)+(2): 2=7-5\mu\Rightarrow\mu=1$
$(1): \lambda=\frac{\mu-5}{2}=-2$
Substituting into $(3): 1+\lambda=-1=-5+4\mu$
The three equations are consistent so the lines intersect, at $X(1,1,-1)$

c

When $\lambda=2$, $\mathbf{r_1}=\begin{pmatrix}9\\-7\\3\end{pmatrix}$ so $Y(9,-7,3)$ lies on L_2

d

$X\hat{Y}Z = 90°, Y\hat{X}Z = 51.7°, XY = \left|\begin{pmatrix} 8 \\ -8 \\ 4 \end{pmatrix}\right| = 12$

Area $XYZ = \frac{1}{2}(XY)(XY \tan 51.7°) = 91.2$

33 **a**

$$\cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{(\sin 2\alpha\cos\alpha + \sin\alpha\cos 2\alpha - 1)}{\sqrt{\sin^2 2\alpha + \cos^2 2\alpha + 1^2}\sqrt{\cos^2\alpha + \sin^2\alpha + 1^2}} = \frac{\sin 3\alpha - 1}{2}$$

b

The vectors are perpendicular when $\cos\theta = 0$

$\sin 3\alpha = 1$

$\alpha = \frac{\pi}{6}$ or $30°$

c

If $\alpha = \frac{7\pi}{6}, \sin 3\alpha = \sin\left(\frac{7\pi}{2}\right) = -1$ so $\cos\theta = -1$; $\theta = \pi$ or $180°$; the two vectors are parallel and run in opposite directions. (ie are antiparallel)

34 **a**

$a = 0$:

$$\begin{cases} 2y + z = 3 & (1) \\ -x + y + 3z = 1 & (2) \\ -2x + y + 2z = k & (3) \end{cases}$$

Eliminating x:

$$\begin{cases} (1): & 2y + z = 3 & (1) \\ 2(2) - (3): & y + 4z = 2 - k & (4) \end{cases}$$

$2(4) - (1): 7z = 1 - k$

$(1): y = \frac{3 - z}{2} = 1 + \frac{k}{2}$

$(2): x = y + 3z - 1 = 3 - \frac{5k}{2}$

There is a unique solution, for each value of k.

b

The original mark scheme for this question suggests using matrices to solve part **b**, which would allow a faster method. In the new curriculum, this method is not available in Analysis, so a somewhat laborious elimination method is likely the most obvious option. Alternatively, given below, a slightly less cumbersome option using normal vectors is possible.

$$\begin{cases} ax + 2y + z = 3 & (1) \\ -x + (a+1)y + 3z = 1 & (2) \\ -2x + y + (a+2)z = k & (3) \end{cases}$$

Eliminating x:

$$\begin{cases} (1) + a(2): & (a^2 + a + 2)y + (1 + 3a)z = 3 + a & (4) \\ 2(2) - (3): & (2a+1)y + (4 - a)z = 2 - k & (5) \end{cases}$$

$2(4) - (1): 7z = 1 - k$

Eliminating z:
$(1+3a)(5)-(4-a)(4)$:
$(2a+1+6a^2+3a+a^3+a^2+2a-4a^2-4a-8)y$
$= (1+3a)(2-k)-(3+a)(4-a)$
$(a^3+3a^2+3a-7)y = a^2+5a-10-(1+3a)k$
$(a-1)(a^2+4a+7)y = a^2+5a-10-(1+3a)k$
This has no solution for y when $a=1$.

c

When $a=1$, the equation in part **b** reduces to
$-4-4k=0$
Therefore when $a=1, k=-1$, the equations are consistent but have no unique solution, but the planes have no common normal; therefore the planes meet in a line.

$$\mathbf{n_1}\times\mathbf{n_2} = \begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-1\\2\\3\end{pmatrix} = \begin{pmatrix}4\\-4\\4\end{pmatrix} = 4\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

The vector $\mathbf{d} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ is the direction vector of this line.

Setting $z=0$ in equation (4) to find a point on the line:
$4y=4 \Rightarrow y=1$
$(1): x+2=3 \Rightarrow x=1$

The line has equation $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$

The vector product approach is shown here:

The planes' normal vectors are

$$\mathbf{n_1} = \begin{pmatrix}a\\2\\1\end{pmatrix}, \mathbf{n_2} = \begin{pmatrix}-1\\a+1\\3\end{pmatrix}, \mathbf{n_3} = \begin{pmatrix}-2\\1\\a+2\end{pmatrix}$$

$\mathbf{n_1}\times\mathbf{n_2} = \begin{pmatrix}5-a\\-1-3a\\a^2+a+2\end{pmatrix}$ which cannot be the zero vector for any value of a

$\mathbf{n_1}\times\mathbf{n_3} = \begin{pmatrix}2a+3\\-a^2-2a-2\\a+4\end{pmatrix}$ which cannot be the zero vector for any value of a

$\mathbf{n_2}\times\mathbf{n_3} = \begin{pmatrix}a^2+3a-1\\a-4\\2a+1\end{pmatrix}$ which cannot be the zero vector for any value of a

If the cross product of two vectors is not the zero vector, then the two vectors are not parallel.

Since none of the normal vectors are parallel, none of the planes can be parallel.

If the planes intersect in a line then all three vector products above must be parallel to the direction vector of the line.

$$\begin{pmatrix}5-a\\-1-3a\\a^2+a+2\end{pmatrix} = k\begin{pmatrix}2a+3\\-a^2-2a-2\\a+4\end{pmatrix}$$

$$\frac{5-a}{2a+3} = \frac{-1-3a}{-a^2-2a-2} = \frac{a^2+a+2}{a+4}$$

Taking the first pair of expressions:

$(5-a)(-2-2a-a^2) = (2a+3)(-1-3a)$

$a^3 - 3a^2 - 8a - 10 = -6a^2 - 11a - 3$

$a^3 + 3a^2 + 3a - 7 = 0$

$(a-1)(a^2+4a+7) = 0$

The only real solution is $a = 1$.

c)

Then $\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{n_1} \times \mathbf{n_3} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{n_2} \times \mathbf{n_3} = 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is the direction of the intersection line.

The planes will intersect in a line if there is a solution, or will form a prism if not.

Finding a point on the intersection of the first two planes:

Fix $z = 0$:

$$\begin{cases} x + 2y = 3 & (1) \\ -x + 2y = 1 & (2) \end{cases}$$

$(1) + (2)$: $4y = 4 \Rightarrow y = 1, x = 1$

For intersection of three planes in a line, require that $(1, 1, 0)$ must lie in the third plane.

Substituting: $-2x + y + (a+2)z = -1 = k$

If $a = 1$ and $k = -1$ then the three planes intersect on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

35 **a**

Other points on the cube are

$A(2,0,0), B(2,2,0), C(0,2,0), D(0,0,2), E(2,0,2), G(0,2,2)$

$\overrightarrow{OM} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \overrightarrow{ON} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \overrightarrow{OP} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

b

$\overrightarrow{MP} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \overrightarrow{MN} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$\overrightarrow{MP} \times \overrightarrow{MN} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

ci

Area of a triangle with side vectors $\mathbf{u}$ and $\mathbf{v}$ is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$

Area $MNP = \frac{1}{2}\left|\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right| = \frac{\sqrt{3}}{2}$

Worked solutions

cii

$$\overrightarrow{AG} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\overrightarrow{AG} \cdot \overrightarrow{MP} = 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = 2 - 2 = 0$ so $\overrightarrow{AG}$ is perpendicular to $\overrightarrow{MP}$

$\overrightarrow{AG} \cdot \overrightarrow{MN} = 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = 2 - 2 = 0$ so $\overrightarrow{AG}$ is perpendicular to $\overrightarrow{MN}$

Therefore $\overrightarrow{AG}$ is perpendicular to the plane MNP

ciii

The scalar product form of plane MNP is therefore $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \overrightarrow{OM} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$-x + y + z = 3$

d

Line AG has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Substituting this into the scalar product form of MNP above to find the value of λ at the point of intersection:

$$\left(\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$$

$-2 + 3\lambda = 3$

$\lambda = \frac{5}{3}$

The intersection point is $\left(\frac{1}{3}, \frac{5}{3}, \frac{5}{3}\right)$

36 **a**

$$\begin{cases} -5 - 3\lambda = 3 + \mu & (1) \\ 1 = \mu & (2) \\ 10 + 4\lambda = -9 + 7\mu & (3) \end{cases}$$

$(2): \mu = 1$

$(1): \lambda = \frac{8 + \mu}{-3} = -3$

Substituting into (3):

$10 + 4\lambda = -2 = -9 + 7\mu$

The system is consistent so the two lines intersect. The point of intersection is $P(4, 1, -2)$

b When $\mu = 2$, $\mathbf{r_2} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$ so Q lies on the line.

c

$\overrightarrow{QM} = \begin{pmatrix} -10-3\lambda \\ -1 \\ 5+4\lambda \end{pmatrix}$. Require $\overrightarrow{QM} \cdot \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = 0$

$30 + 9\lambda + 20 + 16\lambda = 0$

$50 = -25\lambda$

$\lambda = -2$

M has coordinates $(1, 1, 2)$

d

$P\widehat{M}Q = 90°, PM = \left|\begin{matrix} -3 \\ 0 \\ 4 \end{matrix}\right| = 5, QM = \left|\begin{matrix} 4 \\ 1 \\ 3 \end{matrix}\right| = \sqrt{26}$

Therefore the area of triangle PQM is $\frac{1}{2} \times 5 \times \sqrt{26} = \frac{5\sqrt{26}}{2}$

37

$$PQ^2 = \left|\begin{pmatrix} 2+t \\ 1-t \\ 1+t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right|^2$$

$$= (1+t)^2 + (-t)^2 + (t-2)^2$$

$$= 3t^2 - 2t + 5$$

$$= 3\left(t - \frac{1}{3}\right)^2 + \frac{14}{3}$$

This is minimal when $t = \frac{1}{3}$, with value $\sqrt{\frac{14}{3}}$

The minimum distance PQ is $\frac{\sqrt{42}}{3}$ at $t = \frac{1}{3}$

38

The line $\frac{x-3}{1} = \frac{y+1}{2.5} = \frac{z-5}{k}$ has direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 2.5 \\ k \end{pmatrix}$

If the line is within the plane with normal $\mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ then $\mathbf{n} \cdot \mathbf{d} = 0$

$3 + 5 - k = 0$

$k = 8$

Alternatively:

The line has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2k \end{pmatrix}$

Then setting $\lambda = 1$, the point $(5, 4, 5 + 2k)$ lies on the line. Substituting this into the plane equation:

$3x + 2y - z = 2 = 15 + 8 - 5 - 2k$

$-2k = -16$

$k = 8$

39 **a**

The direction of the line of intersection must be perpendicular to both normals.

$\mathbf{n_1} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$

$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} -8 \\ 24 \\ 16 \end{pmatrix} = 8\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

The line has direction vector $\mathbf{d} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

For a point on the line, take $x = 0$ and substitute into both plane equations:

$\begin{cases} -3y + 5z = 12 & (1) \\ \quad y + z = 20 & (2) \end{cases}$

$(1) + 3(2): 8z = 72$

$z = 9 \Rightarrow y = 11$

The line has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

b

Require that Π_3 has direction vectors $\mathbf{d}$ and $\mathbf{n_1}$, so must have normal $\mathbf{n_3} = \mathbf{d} \times \mathbf{n_1}$

$\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 7 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

The line passes through $(0, 11, 9)$ so the plane has scalar product equation $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} =$

$\begin{pmatrix} 0 \\ 11 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$3x + y = 11$

40 **a** for $\mu = 3, \mathbf{r_2} = \begin{pmatrix} 8 \\ 2 \\ 6 \end{pmatrix}$ so $Q(8, 2, 6)$ lies on l_2

b

Both lines pass through $P(2, -1, 0)$ and both direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ have length 3.

Then $PQ = 3|\mathbf{d_2}| = 9$

So $PR = 9$.

$\overrightarrow{PR} = \pm 3\mathbf{d_1} = \pm\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

R has coordinates $(5, -7, 6)$ or $(-1, 5, -6)$

c

The angle bisector will have direction vector which is the sum of the two line direction vectors.

Since R could be on either side of P, the relevant direction vector for l_1 could be $\pm\mathbf{d_1}$.

$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

41 **a**

$\overrightarrow{PC} = \begin{pmatrix} -3 + 2\lambda \\ -\lambda \\ -4 + 2\lambda \end{pmatrix}$

Require $\overrightarrow{PC} \cdot \mathbf{d} = 0$ where $\mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is the direction vector of l.

$-14 + 9\lambda = 0 \Rightarrow \lambda = \frac{14}{9}$

C has coordinates $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$

b

$PC = \left|\frac{1}{9}\begin{pmatrix} 1 \\ -14 \\ -8 \end{pmatrix}\right| = \frac{\sqrt{29}}{3}$

c

$\overrightarrow{PC} = \frac{1}{9}\begin{pmatrix} 1 \\ -14 \\ -8 \end{pmatrix} = \overrightarrow{CQ}$

Q has coordinates $\left(\frac{65}{9}, -\frac{10}{9}, \frac{11}{9}\right)$

42 **a** Area $BCD = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

b $h = |\mathbf{c}| \cos\theta$

c

$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a} \times \mathbf{b}||\mathbf{c}| \cos\theta$

(Since θ is the acute angle between AE and AC, the value must be positive, but $\mathbf{a} \times \mathbf{b}$ may yield the upward of downward vector, depending on the orientation of the triangle, so an absolute value is needed in this formula)

$$\begin{aligned}\text{The volume of a tetrahedron} &= \frac{1}{3} \times \text{base area} \times \text{height} \\ &= \frac{1}{3}\left(\frac{1}{2}|\mathbf{a} \times \mathbf{b}|\right)|\mathbf{c}| \cos\theta \\ &= \frac{1}{6}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|\end{aligned}$$

d

$\mathbf{a} = \overrightarrow{CB} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \overrightarrow{CD} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}, \mathbf{c} = \overrightarrow{CA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{aligned}
\frac{1}{6}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| &= \frac{1}{6}\left|\left(\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right| \\
&= \frac{1}{6}\left|\begin{pmatrix} -7 \\ -1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right| \\
&= \frac{1}{6}|2| \\
&= \frac{1}{3}
\end{aligned}$$

e

$$\begin{aligned}
\text{Area } BCD &= \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \\
&= \frac{1}{2}\left|\begin{pmatrix} -7 \\ -1 \\ 7 \end{pmatrix}\right| \\
&= \frac{1}{2}\sqrt{99}
\end{aligned}$$

$$V = \frac{1}{3}h \times \left(\frac{1}{2}\sqrt{99}\right) = \frac{1}{3}$$

$$h = \frac{2}{\sqrt{99}} = \frac{2\sqrt{11}}{3}$$

f

$$\begin{aligned}
\text{Area } ACD &= \frac{1}{2}|\mathbf{c} \times \mathbf{b}| \\
&= \frac{1}{2}\left|\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}\right| \\
&= \frac{1}{2}\left|\begin{pmatrix} -9 \\ -1 \\ 11 \end{pmatrix}\right| \\
&= \frac{1}{2}\sqrt{203}
\end{aligned}$$

Therefore the distance from B to the face ACD is $\frac{2\sqrt{203}}{203} < \frac{2\sqrt{11}}{3}$.

B is closer to its opposite face.

43 **a**

Solving for an intersection:

$$\begin{cases} -3 + 2\lambda = 5 + \mu & (1) \\ 3 - \lambda = \mu & (2) \\ 18 - 8\lambda = 2 - \mu & (3) \end{cases}$$

$(1) - (2) \Rightarrow -6 + 3\lambda = 5 \Rightarrow \lambda = \frac{11}{3}$

$(2): \mu = 3 - \lambda = -\frac{2}{3}$

Substituting into (3):

$18 - 8\lambda = -\frac{34}{3} \neq 2 - \mu$

The system is inconsistent so the lines do not intersect.

b

When $\lambda = 2, \mathbf{r_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ so $P(1, 1, 2)$ lies on l_1

c

Require that PQ be perpendicular to both line directions $\mathbf{d_1} = \begin{pmatrix} 2 \\ -1 \\ -8 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\mathbf{d_1} \times \mathbf{d_2} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

The line PQ must therefore have equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \nu\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Finding the intersection of PQ with l_2 will give the coordinates of Q:

$$\begin{cases} 1 + 3\nu = 5 + \mu & (4) \\ 1 - 2\nu = \mu & (5) \\ 2 + \nu = 2 - \mu & (6) \end{cases}$$

$(4) - (5): 5\nu = 5 \Rightarrow \nu = 1$

$(5): \mu = -1$

Substituting into (6):

$2 + \nu = 3 = 2 - \mu$

There is a consistent solution, so the two do intersect, at $Q(4, -1, 3)$

d

The plane Π must pass through the midpoint of PQ, point $M(2.5, 0, 2.5)$ and have normal $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

The plane has scalar product equation $\mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0 \\ 2.5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$3x - 2y + z = 10$

e

l_3 will have the same direction vector $\mathbf{d_2}$ but would pass through point P instead of Q

$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \kappa\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

44 **a**
Substituting $x = 2, y = 1, z = 6$ into the plane equation:
$5x - 3y - z = 10 - 3 - 6 = 1$ so P does lie in $5x - 3y - z = 1$

b
$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

The angle between the direction vector and the plane normal $\mathbf{n} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$ is $90° - \theta$

where θ is the angle between the direction vector and the plane.

$\cos(90° - \theta) = \left|\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{|\overrightarrow{PQ}||\mathbf{n}|}\right| = \left|\frac{35}{\sqrt{45}\sqrt{35}}\right| = \frac{\sqrt{7}}{3}$

Then $\sin\theta = \frac{\sqrt{7}}{3}$

c $PQ = \sqrt{45} = 3\sqrt{5}$

d
If R lies on the plane such that RQ is perpendicular to the plane

$P\hat{R}Q = 90°, PQ = 3\sqrt{5}, \sin(R\hat{P}Q) = \frac{\sqrt{7}}{3}$

$QR = PQ\sin(R\hat{P}Q) = \sqrt{35}$

45 **a**
Solving the system for an intersection:

$$\begin{cases} x + 3y + (a-1)z = 1 & (1) \\ 2x + 2y + (a-2)z = 1 & (2) \\ 3x + y + (a-3)z = b & (3) \end{cases}$$

Eliminating x:

$$\begin{cases} 2(1) - (2): & 4y + az = 1 & (4) \\ 3(1) - (3): & 8y + 2az = 3 - b & (5) \end{cases}$$

It is not possible to eliminate y from this pair of simultaneous equations.
Either (4) and (5) are consistent and $(5) = 2(4)$, in which case there is a line intersection or (4) and (5) are inconsistent and there is no common intersection at all, the three planes form a prism.

b If (4) and (5) are consistent then $b = 1$ so that $(5) = 2(4)$

c
The line of intersection has direction perpendicular to the plane normals.

$\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \\ a-1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ a-2 \end{pmatrix} = \begin{pmatrix} a-4 \\ a \\ -4 \end{pmatrix}$

Require that $\mathbf{d}$ is perpendicular to $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

$\begin{pmatrix} a-4 \\ a \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = a - 4 - 3a - 20 = 0$

$-2a = 24$

$a = -12$

46 **a** $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$

b

Substituting the line equation into the plane equation will give the value of λ representing the intersection Q, and $\overrightarrow{PQ} = \lambda\mathbf{n}$

$(\mathbf{p} + \lambda\mathbf{n}) \cdot \mathbf{n} = k$

$\mathbf{p} \cdot \mathbf{n} + \lambda\mathbf{n} \cdot \mathbf{n} = k$

$$\lambda = \frac{(k - \mathbf{p} \cdot \mathbf{n})}{\mathbf{n} \cdot \mathbf{n}} = \frac{(k - \mathbf{p} \cdot \mathbf{n})}{|\mathbf{n}|^2}$$

Then $\overrightarrow{PQ} = \frac{(k - \mathbf{p} \cdot \mathbf{n})}{|\mathbf{n}|^2}\mathbf{n}$

c

Since PQ is perpendicular to the plane, it follows that the shortest distance from P to Π is PQ

$$PQ = \left|\frac{(k - \mathbf{p} \cdot \mathbf{n})}{|\mathbf{n}|^2}\mathbf{n}\right| = \frac{|k - \mathbf{p} \cdot \mathbf{n}|}{|\mathbf{n}|^2}|\mathbf{n}| = \frac{|k - \mathbf{p} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

d

$$\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, k = 22, \mathbf{p} = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

Then the shortest distance is

$$\frac{|k - \mathbf{p} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|22 + 22|}{\sqrt{26}} = \frac{44}{\sqrt{26}} = \frac{22\sqrt{26}}{13}$$

47

Let the meeting point be C

Require that the path of S_2, which is BC, should be of minimal length.

It follows that AC must be perpendicular to BC.

Since the velocity of S_1 is $\begin{pmatrix} 10 \\ 20 \end{pmatrix}$, the velocity of S_2 must be $\pm\begin{pmatrix} -60 \\ 30 \end{pmatrix}$, if it is to be perpendicular and with a magnitude three times as great.

BC has equation $\mathbf{r} = \begin{pmatrix} 70 \\ 30 \end{pmatrix} + \lambda\begin{pmatrix} -60 \\ 30 \end{pmatrix}$, while AC has equation $\mathbf{r} = t\begin{pmatrix} 10 \\ 20 \end{pmatrix}$

These intersect at C:

$$\begin{cases} 70 - 60\lambda = 10t & (1) \\ 30 + 30\lambda = 20t & (2) \end{cases}$$

$2(2) + (1): 130 = 50t \Rightarrow t = 2.6$

$$\lambda = \frac{70 - 10t}{60} = \frac{44}{60}$$

The second ship requires 44 minutes to reach the intersection, and the first ship takes 2 hours and 36 minutes.

If the first ship leaves at $10:00$, they meet at $12:36$ so the second ship leaves port at $11:52$.

48 **ai**

$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}$ by expanding the product
Also, $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}-\mathbf{b}|^2$
$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}$ by expanding the product
Also, $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}+\mathbf{b}|^2$

If $|\mathbf{a}-\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$ so $\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}$
$\Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$
That is, **a** and **b** are perpendicular.

aii

$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}\times\mathbf{b}|\cdot|\mathbf{a}\times\mathbf{b}|$
$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where θ is the angle between the **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector.
Therefore $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}||\mathbf{b}|\sin\theta\,|\mathbf{a}||\mathbf{b}|\sin\theta$
($\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$ by the definition of a unit vector)
$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)$
But $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
So $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$

bi

Triangle area is given by $\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$ where **u** and **v** are vectors of two of the triangle edges.
In ABC, two of the edge vectors are $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c}-\mathbf{a}$

$$\text{Area} = \frac{1}{2}|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$$
$$= \frac{1}{2}|\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{a}|$$
$$= \frac{1}{2}|\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}|$$

(Using that $\mathbf{u}\times\mathbf{u} = \mathbf{0}$ and $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$ for any vectors **u** and **v**).
Reordering the products:
$$\text{Area} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$$

bii

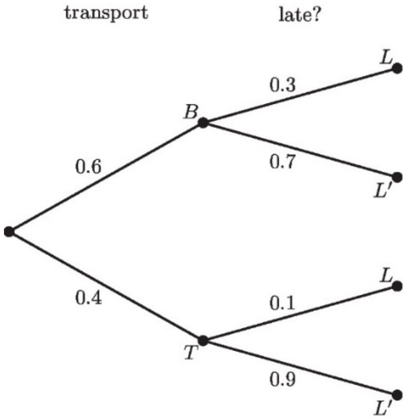
Length AC is $|\mathbf{c}-\mathbf{a}|$
The area of a triangle equals one half the base length multiplied by the altitude to the third vertex.
If the altitude to B is h then
$$\text{Area} = \frac{1}{2}|\mathbf{c}-\mathbf{a}|h$$
$$\frac{1}{2}|\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}| = \frac{1}{2}|\mathbf{c}-\mathbf{a}|h$$
Rearranging:
$$h = \frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{|\mathbf{c}-\mathbf{a}|}$$ as required.

9 Probability

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 9A

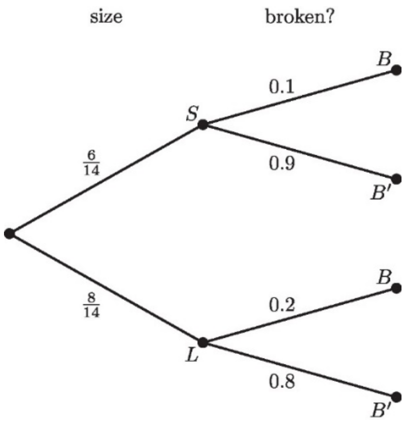
7 **a**



b $P(L') = 0.6 \times 0.7 + 0.4 \times 0.9 = 0.78$

c $P(B|L') = \frac{P(B \cap L')}{P(L')} = \frac{0.6 \times 0.7}{0.78} = 0.538$

8 **a**



b $P(B) = \frac{6}{14} \times 0.1 + \frac{8}{14} \times 0.2 = \frac{22}{140} \approx 0.157$

c $P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{\frac{6}{14} \times 0.1}{0.157} \approx 0.273$

9 **a** $P(S) = P(S|W) \times P(W) + P(S|Y) \times P(Y)$

$= 0.4 \times 0.2 + 0.1 \times 0.8$

$= 0.16$

b $\mathrm{P}(W|S) = \frac{\mathrm{P}(W\cap S)}{\mathrm{P}(S)} = \frac{0.4\times 0.2}{0.16} \approx 0.5$

10 Using the notation B_i is the event that the i^{th} ball is blue, R_i that it is red:

a $\mathrm{P}(\text{different}) = \mathrm{P}(B_1R_2) + \mathrm{P}(R_1B_2)$

$$= \frac{8}{18}\times\frac{10}{17} + \frac{10}{18}\times\frac{8}{17}$$
$$= \frac{80}{153}$$
$$\approx 0.523$$

b $\mathrm{P}(B_2|R_1) = \frac{8}{17}$

c $\mathrm{P}(R_1|\text{different}) = \frac{\mathrm{P}(R_1B_2)}{\mathrm{P}(\text{different})} = \frac{\frac{10}{18}\times\frac{8}{17}}{\frac{80}{153}} = 0.5$

11 a Since there is replacement, the probability is unaffected by the first draw.
$\mathrm{P}(G) = 0.3$

b Since there is replacement, the probabilities of the first and second draw are independent.
$\mathrm{P}(G) = 0.3$

12 a $\mathrm{P}(I) = \mathrm{P}(I|C)\mathrm{P}(C) + \mathrm{P}(I|B)\mathrm{P}(B)$

$$= 0.9\times 0.7 + 0.8\times 0.3$$
$$= 0.87$$

b $\mathrm{P}(C|I) = \frac{\mathrm{P}(C\cap I)}{\mathrm{P}(I)} = \frac{0.9\times 0.7}{0.87} \approx 0.724$

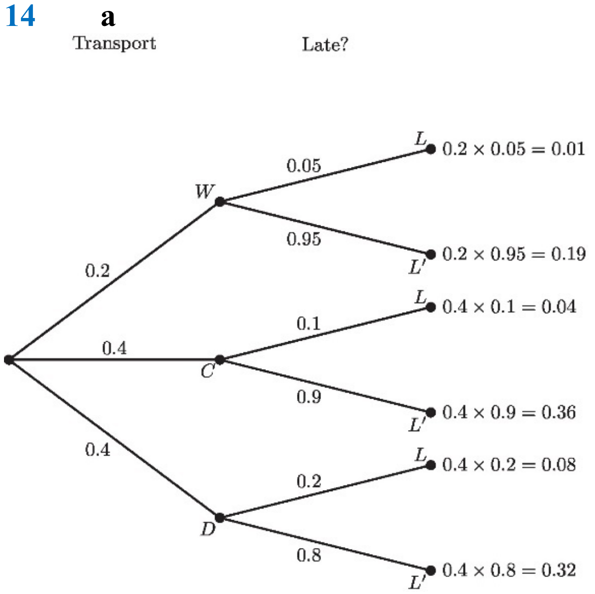
13 a $\mathrm{P}(C) = \mathrm{P}(C|A)\mathrm{P}(A) + \mathrm{P}(C|E)\mathrm{P}(E)$

$$= 0.9\times\frac{20}{38} + 0.93\times\frac{18}{38}$$
$$\approx 0.914$$

b $\mathrm{P}(E|C) = \frac{\mathrm{P}(E\cap C)}{\mathrm{P}(C)}$

$$= \frac{0.93\times\frac{18}{38}}{0.914}$$
$$= 0.482$$

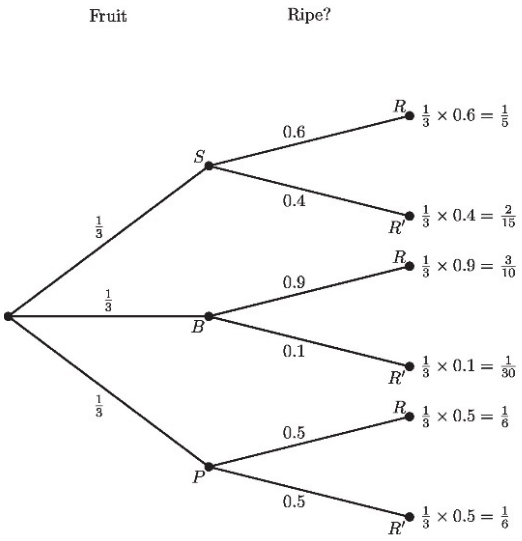
14 a



b $\mathrm{P}(L) = 0.01 + 0.04 + 0.08 = 0.13$

c $\mathrm{P}(W|L) = \frac{\mathrm{P}(W \cap L)}{\mathrm{P}(L)} = \frac{0.01}{0.13} = \frac{1}{13} \approx 0.0769$

15 **a**



b $\mathrm{P}(R) = \frac{1}{5} + \frac{3}{10} + \frac{1}{6} = \frac{2}{3}$

c $\mathrm{P}(S|R) = \frac{\mathrm{P}(S \cap R)}{\mathrm{P}(R)} = \frac{\left(\frac{1}{5}\right)}{\left(\frac{2}{3}\right)} = \frac{3}{10}$

16 $\mathrm{P}(M|P) = \frac{\mathrm{P}(M \cap P)}{\mathrm{P}(P)} = \frac{0.4 \times 0.8}{0.4 \times 0.8 + 0.6 \times 0.6} = \frac{0.32}{0.68} \approx 0.471$

17

$\mathrm{P}(10 \text{ heads}|20 \text{ fair coins}) = \mathrm{P}\big(X = 10 \big| X \sim \mathrm{B}(20, 0.5)\big) \approx 0.176$

$\mathrm{P}(10 \text{ heads}|15 \text{ biased}) = \mathrm{P}(Y = 10 | Y \sim \mathrm{B}\left(15, \frac{2}{3}\right) \approx 0.214$

$\mathrm{P}(\text{fair}|\text{all heads}) = \frac{0.176}{0.176 + 0.214} \approx 0.451$

18

Let X be the outcome of the roll.

$$\mathrm{P}(4 \text{ sides } |X < 6) = \frac{\mathrm{P}(4 \text{ sides} \cap X < 6)}{\mathrm{P}(X < 6)}$$

$$= \frac{\frac{10}{45} \times 1}{\frac{10}{45} \times 1 + \frac{15}{45} \times \frac{5}{6} + \frac{20}{45} \times \frac{5}{8}}$$

$$= \frac{2}{7}$$

19

$$\mathrm{P}(A|S) = \frac{\mathrm{P}(A \cap S)}{\mathrm{P}(S)} = \frac{\frac{1}{3} \times 0.3}{\frac{1}{3} \times 0.3 + \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.2} = \frac{3}{7}$$

20

$$\mathrm{P}(D|+) = \frac{\mathrm{P}(D \cap +)}{\mathrm{P}(+)} = \frac{0.003 \times 0.98}{0.003 \times 0.98 + 0.997 \times 0.1} = 0.0286$$

This highlights the problems of false positive results in tests for rare diseases.

It is a peculiarity of testing that such a test may appear to become more 'accurate' (in the sense that the number of false results decreases) if the prevalence of the disease increases – or more accurately, if the test is more often used on individuals who are infected – even though nothing changes about the test itself!

21

Let + and – be the results of the test (positive and negative) and D be the event that an individual in the population has the disease.

$\mathrm{P}(+|D) = \mathrm{P}(-|D') = 0.9$

Let $\mathrm{P}(D) = p$

Then $\mathrm{P}(+) = \mathrm{P}(+|D)\mathrm{P}(D) + \mathrm{P}(+|D')\mathrm{P}(D')$

$= 0.9p + 0.1(1-p)$

$= 0.1 + 0.8p$

$$\begin{aligned}\mathrm{P}(D|+) &= 0.775\\ &= \frac{\mathrm{P}(D \cap +)}{\mathrm{P}(+)}\\ &= \frac{\mathrm{P}(+|D)\mathrm{P}(D)}{\mathrm{P}(+)}\\ &= \frac{0.9p}{0.1 + 0.8p}\end{aligned}$$

Rearranging:

$0.775(0.1 + 0.8p) = 0.9p$

$$p = \frac{0.0775}{0.9 - 0.8 \times 0.775} = 0.277$$

Approximately 27.7% of the population have the disease.

22

$\mathrm{P}(3\text{ yellow}|A) = \mathrm{P}\big(X = 3 \big| X \sim \mathrm{B}(7, 0.3)\big) = 0.227$

$\mathrm{P}(3\text{ yellow}|B) = \mathrm{P}\big(Y = 3 \big| Y \sim \mathrm{B}(10, 0.3)\big) = 0.267$

$$\begin{aligned}\mathrm{P}(A|3\text{ yellow}) &= \frac{\mathrm{P}(A \cap 3\text{ yellow})}{\mathrm{P}(3\text{ yellow})}\\ &= \frac{0.5 \times 0.227}{0.5 \times 0.227 + 0.5 \times 0.267}\\ &= 0.460\end{aligned}$$

23 **a**

$$\begin{aligned}\mathrm{P}(\text{Same}) &= \mathrm{P}(B_1B_2 \cup W_1W_2)\\ &= \frac{m}{m+n} \times \frac{m-1}{m+n-1} + \frac{n}{m+n} \times \frac{n-1}{m+n-1}\\ &= \frac{m(m-1) + n(n-1)}{(m+n)(m+n-1)}\end{aligned}$$

b

$$P(\text{Blue}|\text{same}) = \frac{P(\text{Blue} \cap \text{same})}{P(\text{same})}$$

$$= \frac{\frac{1}{2} \times \frac{7 \times 6 + 8 \times 7}{15 \times 14}}{\left(\frac{1}{2} \times \frac{5 \times 4 + 15 \times 14}{20 \times 19}\right) + \left(\frac{1}{2} \times \frac{7 \times 6 + 8 \times 7}{15 \times 14}\right)}$$

$$= 0.435$$

Exercise 9B

12 **a**

$$\sum_x P(X = x) = 1 = 0.7 + k$$

$$k = 0.3$$

b

$$E(X) = \sum_x x\, P(X = x) = 1.9$$

c

$$E(X^2) = \sum_x x^2\, P(X = x) = 6.3$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 2.69$$

13 **a**

$$\sum_x P(X = x) = 1 = 0.6 + k$$

$$k = 0.4$$

b

$$E(X) = \sum_x x\, P(X = x) = 2.8$$

$$E(X^2) = \sum_x x^2\, P(X = x) = 9.2$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 1.36$$

$$\text{SD}(X) = \sqrt{\text{Var}(X)} = 1.17$$

14 **a**

$$E(X) = \sum_x x\, P(X = x) = 4.2$$

$$E(X^2) = \sum_x x^2\, P(X = x) = 22.6$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 4.96$$

b

$$E(3X + 1) = 3E(X) + 1 = 13.6$$

$$\text{Var}(3X + 1) = 9\text{Var}(X) = 44.64$$

15 **a**

$$\sum_w \mathrm{P}(W = w) = 1 = k\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8}k$$

$$k = \frac{8}{15}$$

b

$$\mathrm{E}(W) = \sum_w w\,\mathrm{P}(W = w) = 4k = \frac{32}{15}$$

$$\mathrm{E}(W^2) = \sum_w w^2\,\mathrm{P}(W = w) = 8$$

$$\mathrm{Var}(W) = \mathrm{E}(W^2) - \left(\mathrm{E}(W)\right)^2 = \frac{776}{225} \approx 3.45$$

c

$$\mathrm{Var}(2W - 2) = 4\mathrm{Var}(W) \approx 13.8$$

16 **a**

$$\mathrm{E}(X) = \sum_x x\,\mathrm{P}(X = x) = \frac{1 + 6 + 15 + 28}{16} = \frac{25}{8}$$

$$\mathrm{E}(X^2) = \sum_x x^2\,\mathrm{P}(X = x) = \frac{85}{8}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = \frac{85}{8} - \frac{625}{64} = \frac{55}{64}$$

b

$$\mathrm{E}(10X + 3) = 10\mathrm{E}(X) + 3 = 34.25$$
$$\mathrm{Var}(10X + 3) = 100\mathrm{Var}(X) \approx 85.9$$

17 **a**

$$\mathrm{E}(X) = \sum_x x\,\mathrm{P}(X = x) = 4.1$$

$$\mathrm{E}(X^2) = \sum_x x^2\,\mathrm{P}(X = x) = 19.9$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = 3.09$$

b

y	7	10	16	25
$\mathrm{P}(Y = y)$	0.2	0.3	0.4	0.1

c

$$\mathrm{E}(Y) = \sum_y y\,\mathrm{P}(Y = y) = 13.3$$

$$\mathrm{E}(Y^2) = \sum_y y^2\,\mathrm{P}(Y = y) = 204.7$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2 = 27.81 = 9\mathrm{Var}(X)$$

18 **a**

v	2	4	6	8
$\mathrm{P}(V = v)$	0.1	0.2	0.3	0.4

$$\mathrm{E}(V) = \sum_v v\,\mathrm{P}(V = v) = 6$$

$$\mathrm{E}(V^2) = \sum_v v^2\, \mathrm{P}(V = v) = 40$$

$$\mathrm{Var}(V) = \mathrm{E}(V^2) - \left(\mathrm{E}(V)\right)^2 = 4$$

b

$W = 5 - V$

W	3	1	–1	–3
$\mathrm{P}(W = w)$	0.1	0.2	0.3	0.4

c

$$\mathrm{E}(W) = \sum_w w\, \mathrm{P}(W = w) = -1$$

$$\mathrm{E}(W^2) = \sum_w w^2\, \mathrm{P}(W = w) = 5$$

$$\mathrm{Var}(W) = \mathrm{E}(W^2) - \left(\mathrm{E}(W)\right)^2 = 4 = \mathrm{Var}(V)$$

19

$B = 5 - 0.5A$

$\mathrm{E}(B) = 5 - 0.5\mathrm{E}(A) = 3.1$

$\mathrm{Var}(B) = 0.25\mathrm{Var}(A) = 0.3$

20

$V = 4 - 0.4U$

$\mathrm{E}(V) = 4 - 0.4\mathrm{E}(U) = -6$

$\mathrm{Var}(V) = 0.16\mathrm{Var}(U) = 2.56$

21 **a**

Y	0	1	2	3	4
$\mathrm{P}(Y = y)$	0.1	0.2	0.2	0.3	0.2

b

$$\mathrm{E}(Y) = \sum_y y\, \mathrm{P}(Y = y) = 2.3$$

$$\mathrm{E}(Y^2) = \sum_y y^2\, \mathrm{P}(Y = y) = 6.9$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2 = 1.61$$

c

$X = 3 + 100Y$

$\mathrm{E}(X) = 3 + 100\mathrm{E}(Y) = 233$

$\mathrm{Var}(X) = 10000\mathrm{Var}(Y) = 16100$

22 **a** $\mathrm{X} \sim B\left(3, \frac{1}{2}\right)$

X	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b $\mathrm{E}(X) = 3 \times \frac{1}{2} = 1.5$

c

$$\mathrm{E}(X^2) = \sum_x x^2\, \mathrm{P}(X = x) = 3$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = 0.75$$

Then $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \dfrac{\sqrt{3}}{2}$

Worked solutions

23

X	1	2	3
$P(X = x)$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$

$$E(X) = \sum_x x\, P(X = x) = \frac{11}{6}$$

$$E(X^2) = \sum_x x^2\, P(X = x) = \frac{23}{6}$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = \frac{17}{36} \approx 0.472$$

If $Y = X_1 + X_2$

$$E(Y) = 2E(X) = \frac{11}{3}$$

$$\text{Var}(Y) = 2\text{Var}(X) = \frac{17}{18}$$

$$\text{SD}(Y) = \sqrt{\text{Var}(Y)} \approx 0.972$$

24 **a**

$$E(W) = 2 = \sum_w w\, P(W = w) = 0.4 + 0.6 + 3a + 4b$$

$$3a + 4b = 1 \qquad (1)$$

b

$$\sum_w P(W = w) = 1 = 0.7 + a + b$$

$$a + b = 0.3 \qquad (2)$$

c

$$(1) - 3(2): b = 0.1 \Rightarrow a = 0.2$$

$$E(W^2) = \sum_w w^2\, P(W = w) = 5$$

$$\text{Var}(W) = E(W^2) - \left(E(W)\right)^2 = 1$$

25

$$E(X) = 1.5 = \sum_x x\, P(X = x) = p + 2q + 0.6$$

$$p + 2q = 0.9 \qquad (1)$$

$$\sum_x P(X = x) = 1 = 0.3 + p + q$$

$$p + q = 0.7 \qquad (2)$$

$$(1) - (2): q = 0.2 \Rightarrow p = 0.5$$

$$E(X^2) = \sum_x x^2\, P(X = x) = 3.1$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 0.85$$

26 **a**

$X \sim \mathrm{B}\left(30, \frac{1}{6}\right)$

$\mathrm{E}(X) = 30 \times \frac{1}{6} = 5$

$\mathrm{Var}(X) = 30 \times \frac{1}{6} \times \frac{5}{6} = \frac{25}{6}$

b

$T = 10X - c$

$\mathrm{E}(T) = 10\mathrm{E}(X) - c = 50 - c$

$\mathrm{Var}(T) = 100\mathrm{Var}(X) = \frac{1250}{3}$

c

The game is fair if the expectation is zero; $c = 50$ cents

27

$\mathrm{E}(S) = \sum_s s\,\mathrm{P}(S = s) = \frac{1}{6}(1 + 2 + 5 + 8 + 10 + c) = \frac{26 + c}{6}$

$\mathrm{E}(S^2) = \sum_s s^2\,\mathrm{P}(S = s) = \frac{1}{6}(1 + 4 + 25 + 64 + 100 + c^2) = \frac{194 + c^2}{6}$

$\mathrm{Var}(S) = \mathrm{E}(S^2) - \left(\mathrm{E}(S)\right)^2 = \frac{1}{36}(6(194 + c^2) - (26 + c)^2) = \frac{185}{9}$

$6(194 + c^2) - (26 + c)^2 = 740$

$488 - 52c + 5c^2 = 740$

$5c^2 - 52c - 252 = 0$

$c = \frac{52 \pm \sqrt{52^2 + 5040}}{10} = 14 \text{ or} - 3.6$

Since $c > 10$, the value is 14.

28

$\mathrm{E}(X) = \sum_x x\,\mathrm{P}(X = x) = q + 1$

$\mathrm{E}(X^2) = \sum_x x^2\,\mathrm{P}(X = x) = q + 2.6$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = 2.6 + q - (q + 1)^2 = 0.85$

$q^2 + q - 0.75 = 0$

$(q + 1.5)(q - 0.5) = 0$

$q = -1.5$ (reject) or $q = 0.5$

Then $\mathrm{E}(X) = 1 + q = 1.5$

Exercise 9C

Except in non-calculator questions, it is expected that students will calculate probabilities from the integrals of probability generating functions using technology rather than integral calculus. However, to assist those wishing to practice techniques learned in the Chapter 10, many have been given full worked algebraic solutions.

22 **a**

$$P\left(\frac{\pi}{16} \le X \le \frac{\pi}{8}\right) = \int_{\frac{\pi}{16}}^{\frac{\pi}{8}} \frac{2}{\ln 2} \tan x \; dx$$

$$= \left[\frac{2}{\ln 2} \ln(\sec x)\right]_{\frac{\pi}{16}}^{\frac{\pi}{8}}$$

$$= 0.172$$

b

$$P\left(X > \frac{3\pi}{16}\right) = \int_{\frac{3\pi}{16}}^{\frac{\pi}{4}} \frac{2}{\ln 2} \tan x$$

$$= \left[\frac{2}{\ln 2} \ln(\sec x)\right]_{\frac{3\pi}{16}}^{\frac{\pi}{4}}$$

$$= 0.467$$

23 **a**

$$P(0.5 < Y < 1) = \int_{0.5}^{1} \frac{3}{4} y(2-y) \; dy$$

$$= \left[\frac{3}{4}y^2 - \frac{1}{4}y^3\right]_{0.5}^{1}$$

$$= \left(\frac{3}{4} - \frac{1}{4}\right) - \left(\frac{3}{16} - \frac{1}{32}\right)$$

$$= \frac{11}{32}$$

b

$$P\left(Y > \frac{2}{3}\right) = \int_{\frac{2}{3}}^{2} \frac{3}{4} y(2-y) \; dy$$

$$= \left[\frac{3}{4}y^2 - \frac{1}{4}y^3\right]_{\frac{2}{3}}^{2}$$

$$= (3-2) - \left(\frac{1}{3} - \frac{2}{27}\right)$$

$$= \frac{20}{27}$$

24 **a**

$$\int_{-\infty}^{\infty} f(x) \; dx = 1 = \int_{3}^{8} k e^{-x} \; dx$$

$$k[-e^{-x}]_3^8 = 1$$

$$k = \frac{1}{e^{-3} - e^{-8}} \approx 20.2$$

b

$$\mathrm{P}(X > 5) = \int_5^8 k\mathrm{e}^{-x}\ \mathrm{d}x$$
$$= [-k\mathrm{e}^{-x}]_5^8$$
$$= \frac{\mathrm{e}^{-5} - \mathrm{e}^{-8}}{\mathrm{e}^{-3} - \mathrm{e}^{-8}}$$
$$\approx 0.129$$

c

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\ \mathrm{f}(x)\ \mathrm{d}x$$
$$= \int_3^8 kx\mathrm{e}^{-x}\ \mathrm{d}x$$
$$= 3.97\ (\mathrm{GDC})$$

This function can be integrated directly using the method of integration by parts, described in Chapter 10. However, unless the question explicitly requires an analytical method, remember that evaluation of an integral using your calculator is acceptable, and even preferred.

25 **a**

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = 1 = \int_0^3 kx^2\ \mathrm{d}x$$
$$k\left[\frac{1}{3}x^3\right]_0^3 = 1$$
$$k = \frac{1}{9}$$

b

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\ \mathrm{f}(x)\ \mathrm{d}x$$
$$= \int_0^3 kx^3\ \mathrm{d}x$$
$$= \left[\frac{1}{36}x^4\right]_0^3$$
$$= \frac{9}{4}$$

26

Median m is such that $\int_{-\infty}^{m} \mathrm{f}(x)\ \mathrm{d}x = \frac{1}{2}$

$$\int_0^m (2 - 2x)\ \mathrm{d}x = [2x - x^2]_0^m = 2m - m^2 = \frac{1}{2}$$
$$2m^2 - 4m + 1 = 0$$
$$m = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{\sqrt{2}}{2}$$
$$m < 1 \text{ so } m = 1 - \frac{\sqrt{2}}{2} \approx 0.293$$

27 **a**

$$P(T > 1) = \int_1^2 \frac{t}{2}\, dt$$
$$= \left[\frac{1}{4}t^2\right]_1^2$$
$$= \frac{3}{4}$$

b

Median m is such that $\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2}$

$$\int_0^m \frac{1}{2}t\, dt = \left[\frac{1}{4}t^2\right]_0^m$$
$$= \frac{m^2}{4} = \frac{1}{2}$$

$m = \sqrt{2}$ (selecting positive root since $0 < m < 2$)

28 **a**

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$
$$= \int_0^{\pi^2} \frac{y}{2\pi} \sin\sqrt{y}\, dy$$
$= 3.87$ (GDC)

b

$$E(Y^2) = \int_{-\infty}^{\infty} y^2\, f(y)\, dy$$
$$= \int_0^{\pi^2} \frac{y^2}{2\pi} \sin\sqrt{y}\, dy$$
$= 20.0$ (GDC)

$\text{Var}(Y) = E(Y^2) - \left(E(Y)\right)^2 = 5.043$

$\text{SD}(Y) = \sqrt{\text{Var}(Y)} = 2.25$

c

Median m is such that $\int_{-\infty}^{m} f(y)\, dy = \frac{1}{2}$

$$\int_0^m \frac{1}{2\pi} \sin\sqrt{y}\, dy = \frac{1}{2}$$

From GDC, $m \approx 3.63$

Again, using methods of substitution and integration by parts seen in Chapter 10, this integral can be calculated analytically:

Let $y = u^2$ so that $dy = 2u\, du$

$$\int_0^{\sqrt{m}} \frac{u}{\pi} \sin u\, du = \left[-\frac{u}{\pi}\cos u\right]_0^{\sqrt{m}} + \int_0^{\sqrt{m}} \frac{1}{\pi}\cos u\, du$$
$$= -\frac{\sqrt{m}}{\pi}\cos\sqrt{m} + \frac{1}{\pi}\sin\sqrt{m} = \frac{1}{2}$$

However, this final result cannot be solved algebraically and would require a calculator solution to find $m \approx 3.63$.

d

From GDC, the probability density function has its maximum at $x = 2.47$

29 **a**

$E(X) = 0$ since the probability density function is symmetrical about 0.

b

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \, f(x) \, dx$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{2} \cos x \, dx$$
$$\approx 0.467$$

$$\mathrm{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 0.467$$

Using the integration by parts method from Chapter 10, the variance can be calculated exactly:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{2} \cos x \, dx = \left[\frac{x^2}{2} \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \, dx$$
$$= \left[\frac{x^2}{2} \sin x + x \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$
$$= \left[\frac{x^2}{2} \sin x + x \cos x - \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{\pi^2}{4} - 2$$

30 **a**

$$P(X > 0.5) = \int_{0.5}^{1} \frac{e}{2e - 5} x^2 e^{-x} \, dx$$
$$= 0.821 \text{ (GDC)}$$

b

Maximum on the PDF within the given domain is at $x = 1$

c

$$E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$
$$= \int_{0}^{1} \frac{e}{2e - 5} x^3 e^{-x} \, dx$$
$$= 0.709 \text{ (GDC)}$$

31 **a**

$$\int_{-\infty}^{\infty} f(y) \, dy = 1 = \int_{1}^{3} \frac{1}{ky} \, dy = \frac{1}{k} \ln 3$$

$k = \ln 3$

b

Median m is such that $\int_{-\infty}^{m} f(y) \, dy = \frac{1}{2}$

$$\int_{1}^{m} \frac{1}{y \ln 3} \, dy = \frac{1}{\ln 3} \ln m = \frac{1}{2}$$

$m = \sqrt{3}$

Worked solutions

32 **a**

$f'(x) = 4 - 12x^2 f'(x) = 0$ at a local maximum of the pdf; this occurs at $x = \pm\frac{1}{\sqrt{3}}$

$f''(x) = -24x < 0$ at $x = \frac{1}{\sqrt{3}}$, so this is a maximum of the pdf and so is the mode.

b

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}(x)\;\mathrm{d}x$$
$$= \int_0^1 4x^2 - 4x^4\;\mathrm{d}x$$
$$= \left[\frac{4}{3}x^3 - \frac{4}{5}x^5\right]_0^1$$
$$= \frac{4}{3} - \frac{4}{5}$$
$$= \frac{8}{15}$$

33 **a**

$$\mathrm{E}(T) = \int_{-\infty}^{\infty} t\,\mathrm{f}(t)\;\mathrm{d}t$$
$$= \int_{10}^{30} \frac{3}{40000} t(30-t)(t-10)^2\;\mathrm{d}t$$
$$= 22\ (\mathrm{GDC})$$

b

$$\mathrm{P}(T > 25) = \int_{25}^{30} \mathrm{f}(t)\;\mathrm{d}t = 0.262$$

Let X be the number of students taking more than 25 minutes. Assuming independence between students, $\mathrm{E}(X) = 30 \times 0.262 \approx 7.85 \approx 8$ students

34 **ai**

$$\mathrm{P}(5 < X < 10) = \int_5^{10} \mathrm{f}(x)\;\mathrm{d}x$$
$$= \left[-\frac{\mathrm{e}^4}{(\mathrm{e}^4 - 1)}\mathrm{e}^{-\frac{x}{5}}\right]_5^{10}$$
$$= \frac{\mathrm{e}^3 - \mathrm{e}^2}{\mathrm{e}^4 - 1}$$
$$\approx 0.237$$

aii

$$\mathrm{P}(X < 10) = \int_0^{10} \mathrm{f}(x)\;\mathrm{d}x$$
$$= \left[-\frac{\mathrm{e}^4}{(\mathrm{e}^4 - 1)}\mathrm{e}^{-\frac{x}{5}}\right]_0^{10}$$
$$= \frac{\mathrm{e}^4 - \mathrm{e}^2}{\mathrm{e}^4 - 1}$$
$$\approx 0.881$$

b

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}(x)\ \mathrm{d}x$$
$$= \int_0^{20} \frac{x\mathrm{e}^4}{5(\mathrm{e}^4 - 1)} \mathrm{e}^{-\frac{x}{5}}\ \mathrm{d}x$$
$$\approx 4.63 \text{ minutes (GDC)}$$

When you have completed Chapter 10, use integration by parts to show that the exact value is

$$\mathrm{E}(X) = \frac{5\mathrm{e}^4 - 25}{\mathrm{e}^4 - 1}$$

35 **a** $\mathrm{f}(x)$ is zero or positive throughout its domain

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = \int_0^k \frac{2x}{k^2}\ \mathrm{d}x$$
$$= \left[\frac{x^2}{k^2}\right]_0^k$$
$$= 1$$

Since both these conditions are met, $\mathrm{f}(x)$ is a valid pdf for $k > 0$.

b

Median m is such that $\int_{-\infty}^{m} \mathrm{f}(x)\ \mathrm{d}x = \frac{1}{2}$

$$\int_0^m \frac{2x}{k^2}\ \mathrm{d}x = \frac{m^2}{k^2} = \frac{1}{2}$$

$m = \frac{k}{\sqrt{2}}$ (Selecting positive root since $0 \le m \le k$)

c

$$\mathrm{P}(X \ge c) = \int_c^k \frac{2x}{k^2}\ \mathrm{d}x = \frac{k^2 - c^2}{k^2} = 0.19$$
$$c^2 = 0.81k^2$$
$$c = 0.9k$$

36 **a**

Require that $\int_{-\infty}^{\infty} \mathrm{f}(t)\ \mathrm{d}t = 1$

$$\int_k^{k+1} \frac{1}{1+t}\ \mathrm{d}t = [\ln|1+t|]_k^{k+1} = \ln\left|\frac{k+2}{k+1}\right| = 1$$
$$\frac{k+2}{k+1} = \mathrm{e}$$
$$k = \frac{\mathrm{e} - 2}{1 - \mathrm{e}} \approx -0.418$$

b

$$\begin{aligned}\mathrm{E}(T) &= \int_{-\infty}^{\infty} t\,\mathrm{f}(t)\ \mathrm{d}t\\ &= \int_{k}^{k+1} \frac{t}{t+1}\ \mathrm{d}t\\ &= \int_{k}^{k+1} 1-\frac{1}{t+1}\ \mathrm{d}t\\ &= [t]_k^{k+1}-1\\ &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{E}(T^2) &= \int_{-\infty}^{\infty} t^2\,\mathrm{f}(t)\ \mathrm{d}t\\ &= \int_{k}^{k+1} \frac{t^2}{t+1}\ \mathrm{d}t\\ &= \int_{k}^{k+1} t-1+\frac{1}{t+1}\ \mathrm{d}t\\ &= \left[\frac{t^2}{2}-t\right]_k^{k+1}+1\\ &= \frac{(k+1)^2}{2}-(k+1)-\left(\frac{k^2}{2}-k\right)+1\\ &= -k+\frac{1}{2}\\ &= \frac{\mathrm{e}-3}{2(1-\mathrm{e})}\approx 0.0820\end{aligned}$$

37 **a**

$\mathrm{P}(X>a)=\int_a^3 \frac{x^2}{9}\ \mathrm{d}x=\left[\frac{1}{27}x^3\right]_a^3=1-\frac{a^3}{27}$

Require $\frac{a^3}{27}=0.95$

$a=2.95$

b

Lower quartile L and upper quartile U are such that

$\mathrm{P}(X>L)=0.75=1-\frac{L^3}{27}$ and $\mathrm{P}(X>U)=0.25=1-\frac{U^3}{27}$

$L=1.89, U=2.73$

Interquartile range $U-L=0.836$

38

$\mathrm{P}(X>a)=\int_a^{\mathrm{e}} \frac{1}{x}\ \mathrm{d}x=[\ln|x|]_a^{\mathrm{e}}=1-\ln a$

Lower quartile L and upper quartile U are such that

$\mathrm{P}(X>L)=0.75=1-\ln L$ and $\mathrm{P}(X>U)=0.25=1-\ln U$

$L=\mathrm{e}^{0.25}, U=\mathrm{e}^{0.75}$

Interquartile range $U-L\approx 0.833$

39 **a**

Require $\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = 1$

$$\int_0^{10} kx\ \mathrm{d}x + \int_{10}^{20} k(20-x)\ \mathrm{d}x = \left[\frac{1}{2}kx^2\right]_0^{10} + \left[20kx - \frac{1}{2}kx^2\right]_{10}^{20}$$
$$= 50k + \left(400k - 200k - (200k - 50k)\right)$$
$$= 100k$$

$$k = \frac{1}{100}$$

b

$$\mathrm{P}(X > 15) = \int_{15}^{20} k(20-x)\ \mathrm{d}x = \left[20kx - \frac{1}{2}kx^2\right]_{15}^{20} = 12.5k = \frac{1}{8}$$

c

$$\mathrm{P}(X > a) = 0.1 = \left[20kx - \frac{1}{2}kx^2\right]_a^{20} = \left(200 - 20a + \frac{1}{2}a^2\right)k$$

$a^2 - 40a + 400 = 20$

$a^2 - 40a + 380 = 0$

$a = 20 \pm \sqrt{20}$

Taking the negative root, since $a < 20$, $a = 20 - \sqrt{20} \approx 15.5$

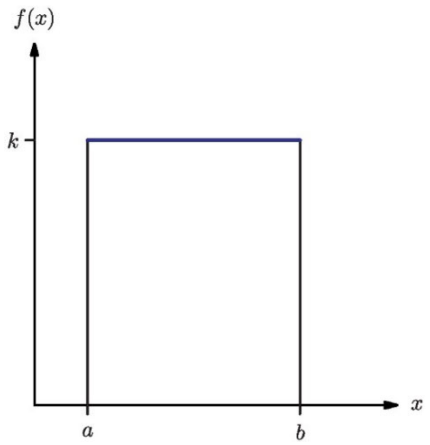
d

$\mathrm{E}(X) = 10$ by symmetry

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\ \mathrm{f}(x)\ \mathrm{d}x$$
$$= \int_0^{10} \frac{x^3}{100}\ \mathrm{d}x + \int_{10}^{20} \frac{20x^2 - x^3}{100}\ \mathrm{d}x$$
$$= \left[\frac{x^4}{400}\right]_0^{10} + \left[\frac{80x^3 - 3x^4}{1200}\right]_{10}^{20}$$
$$= \frac{10000}{400} + \frac{64000 - 480000 - 80000 + 30000}{1200}$$
$$= \frac{350}{3}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = \frac{50}{3} \approx 16.7$$

40 **a**



b

By symmetry, if $\mathrm{f}(x)$ is a pdf, then $\mathrm{E}(X) = \frac{a+b}{2}$

c

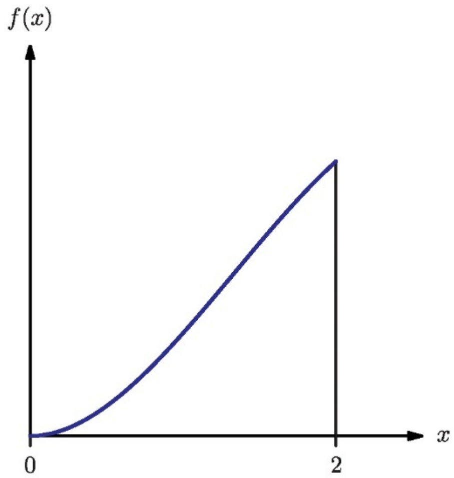
For $\mathrm{f}(x)$ to be a pdf, require that $\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = 1$

$$k = \frac{1}{b-a}$$

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2\ \mathrm{f}(x)\ \mathrm{d}x \\
&= \int_a^b \frac{x^2}{b-a}\ \mathrm{d}x \\
&= \left[\frac{x^3}{3(b-a)}\right]_a^b \\
&= \frac{b^3-a^3}{3(b-a)} \\
&= \frac{b^2+ab+a^2}{3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 \\
&= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{12}(4a^2+4ab+4b^2-3a^2-6ab-3b^2) \\
&= \frac{1}{12}(a^2-2ab+b^2) \\
&= \frac{(a-b)^2}{12}
\end{aligned}$$

41 **a**



b

The mode is the upper limit of the pdf, since the function is increasing throughout the domain.

Mode is 2

c

$$\text{E}(X) = \int_{-\infty}^{\infty} x\,\text{f}(x)\ \text{d}x$$
$$= \int_0^2 \frac{3}{20}(4x^3 - x^4)\ \text{d}x$$
$$= \left[\frac{3}{100}(5x^4 - x^5)\right]_0^2$$
$$= \frac{3}{100}(80 - 32)$$
$$= \frac{36}{25}$$

42 **a**

For $\text{f}(y)$ to be a pdf, require that $\int_{-\infty}^{\infty} \text{f}(y)\ \text{d}y = 1$

$$\int_{-k}^{k} ay^2\ \text{d}y = \left[\frac{a}{3}y^3\right]_{-k}^{k} = \frac{2ak^3}{3} = 1$$
$$a = \frac{3}{2k^3}$$

b

$\text{E}(Y) = 0$ by symmetry

$$\text{E}(Y^2) = \int_{-\infty}^{\infty} y^2\,\text{f}(y)\ \text{d}y$$
$$= \int_{-k}^{k} ay^4\ \text{d}y$$
$$= \left[\frac{1}{5}ay^5\right]_{-k}^{k}$$
$$= \frac{2}{5}ak^5$$
$$= \frac{3}{5}k^2$$
$$\text{Var}(Y) = \text{E}(Y^2) - \left(\text{E}(Y)\right)^2$$
$$= \frac{3}{5}k^2 = 5$$
$$k^2 = \frac{25}{3}$$
$$k = \frac{5\sqrt{3}}{3}$$

43

For $\text{f}(x)$ to be a pdf, require that $\int_{-\infty}^{\infty} \text{f}(x)\ \text{d}x = 1$

$$\int_{-3}^{3} k(9 - x^2)\,dx = \left[\frac{k}{3}(27x - x^3)\right]_{-3}^{3} = 36k = 1$$
$$k = \frac{1}{36}$$

By symmetry, $\text{E}(X) = 0$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\, \mathrm{f}(x)\ \mathrm{d}x$$
$$= \int_{-3}^{3} k(9x^2 - x^4)\ \mathrm{d}x$$
$$= \left[\frac{k}{5}(15x^3 - x^5)\right]_{-3}^{3}$$
$$= \frac{324k}{5}$$
$$= \frac{9}{5}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2$$
$$= \frac{9}{5}$$

Then $\mathrm{SD}(X) = \frac{3\sqrt{5}}{5}$

$$\mathrm{P}\left(-\frac{6\sqrt{5}}{5} < X < \frac{6\sqrt{5}}{5}\right) = \int_{-\frac{6\sqrt{5}}{5}}^{\frac{6\sqrt{5}}{5}} k(9 - x^2)\, dx = \left[\frac{k}{3}(27x - x^3)\right]_{-\frac{6\sqrt{5}}{5}}^{\frac{6\sqrt{5}}{5}} = \frac{11\sqrt{5}}{25}$$
$$\approx 0.984$$

44 **a**

$$\mathrm{P}(Y < 5) = 1 - \mathrm{P}(Y > 5)$$
$$= 1 - \int_{5}^{6} \frac{1}{27}(6y - y^2)\ \mathrm{d}y$$
$$= 1 - \left[\frac{1}{81}(9y^2 - y^3)\right]_{5}^{6}$$
$$= 1 - \left(4 - \frac{8}{3}\right) + \left(\frac{225}{81} - \frac{125}{81}\right)$$
$$= \frac{73}{81} \approx 0.901$$

b

From the graph, the median will be greater than 3, so use the upper part of the distribution:

Median m is such that $\int_{m}^{\infty} \mathrm{f}(y)\ \mathrm{d}y = \frac{1}{2}$

$$\int_{m}^{6} \frac{1}{27}(6y - y^2)\ \mathrm{d}y = \frac{1}{2}$$
$$= \left[\frac{1}{81}(9y^2 - y^3)\right]_{m}^{6}$$
$$= \left(4 - \frac{8}{3}\right) + \left(\frac{9m^2}{81} - \frac{m^3}{81}\right)$$
$$= \frac{108 + m^3 - 9m^2}{81}$$

$2m^3 - 18m^2 + 216 = 81$

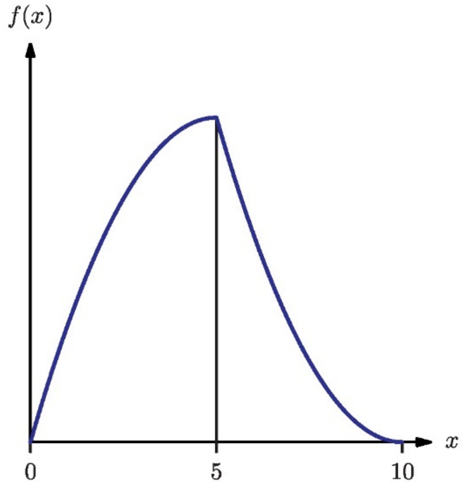
$2m^3 - 18m^2 + 135 = 0$

From GDC, this has solution $m = 3.50$

c

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} \mathrm{f}(y)\ \mathrm{d}y$$
$$= \int_0^3 \frac{y^3}{27}\ \mathrm{d}y + \int_3^6 \frac{6y^2 - y^3}{27}\ \mathrm{d}y$$
$$= \left[\frac{y^4}{108}\right]_0^3 + \left[\frac{8y^3 - y^4}{108}\right]_3^6$$
$$= \frac{3}{4} + \left(16 - 12 - \left(2 - \frac{3}{4}\right)\right)$$
$$= \frac{7}{2}$$

45 **a**



b

$$\mathrm{P}(T < 5) = \int_0^5 \frac{10t - t^2}{125}\ \mathrm{d}t$$
$$= \left[\frac{15t^2 - t^3}{375}\right]_0^5$$
$$= \frac{2}{3}$$

c

Median m is such that $\int_0^m \mathrm{f}(t)\ \mathrm{d}t = \frac{1}{2}$

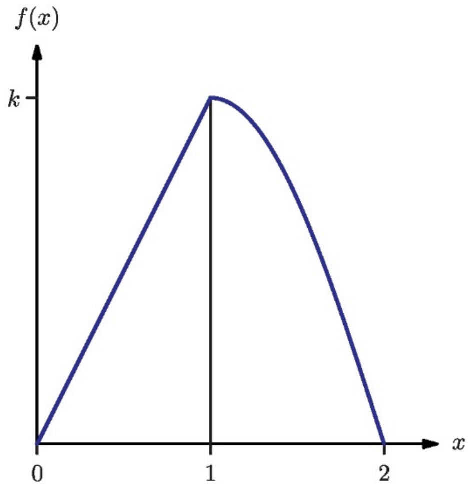
$$\int_o^m \frac{10t - t^2}{125}\ \mathrm{d}t = \left[\frac{15t^2 - t^3}{375}\right]_0^m = \frac{15m^2 - m^3}{375} = \frac{1}{2}$$
$$2(15m^2 - m^3) = 375$$
$$2m^3 - 30m^2 + 375 = 0$$

46 **a**

For $\mathrm{f}(t)$ to be a pdf, require that $\int_{-\infty}^{\infty} \mathrm{f}(t)\ \mathrm{d}t = 1$

$$\int_0^1 kt\ \mathrm{d}t + \int_1^2 k\sin\left(\frac{\pi t}{2}\right)\ \mathrm{d}t = \frac{k}{2} + \left[-\frac{2k}{\pi}\cos\left(\frac{\pi t}{2}\right)\right]_1^2 = \frac{k}{2} + \frac{2k}{\pi} = \frac{k(\pi + 4)}{2\pi}$$
$$k = \frac{2\pi}{\pi + 4}$$

b



c

From the graph, the median will be greater than 1, so use the upper part of the distribution:

Median m is such that $\int_m^\infty \mathrm{f}(t)\ \mathrm{d}t = \frac{1}{2}$

$$\int_m^2 k\sin\left(\frac{\pi t}{2}\right)\ \mathrm{d}t = \left[-\frac{2k}{\pi}\cos\left(\frac{\pi t}{2}\right)\right]_m^2 = \frac{2k}{\pi}\left(1+\cos\left(\frac{\pi m}{2}\right)\right) = \frac{1}{2}$$

$$m = \frac{2}{\pi}\arccos\left(\frac{\pi}{4k} - 1\right) = \frac{2}{\pi}\arccos\left(\frac{\pi - 4}{8}\right) \approx 1.07$$

Mixed Practice

1 **a**

$$\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X = x) = 3.2$$

$$\mathrm{E}(X^2) = \sum_x x^2\ \mathrm{P}(X = x) = 11.8$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = 1.56$$

b

$$\mathrm{E}(2 - 3X) = 2 - 3\mathrm{E}(X) = -7.6$$
$$\mathrm{Var}(2 - 3X) = 9\mathrm{Var}(X) = 14.04$$

2 **a**

$$\sum_v \mathrm{P}(V = v) = 1 = 6p + 0.4$$

$p = 0.1$

b

$$\mathrm{E}(V) = \sum_v v\ \mathrm{P}(V = v) = 4.2$$

$$\mathrm{E}(V^2) = \sum_v v^2\ \mathrm{P}(V = v) = 22.2$$

$$\mathrm{Var}(V) = \mathrm{E}(V^2) - \left(\mathrm{E}(V)\right)^2 = 4.56$$

$$SD(V) = \sqrt{Var(V)} = 2.14$$

c

$\mathrm{E}(10 - V) = 10 - \mathrm{E}(V) = 5.8$

$\mathrm{SD}(10 - V) = \mathrm{SD}(V) = 2.14$

3 **a**

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = 1 = \int_0^6 k(6x - x^2)\ \mathrm{d}x$$

$$k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 36k = 1$$

$$k = \frac{1}{36}$$

b

$$\mathrm{P}(X > 2) = \int_2^6 k(6x - x^2)\ \mathrm{d}x$$

$$= k\left[3x^2 - \frac{x^3}{3}\right]_2^6$$

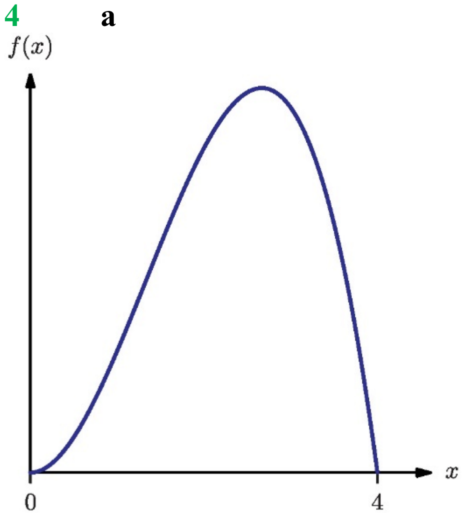
$$= \frac{36 - \left(12 - \frac{8}{3}\right)}{36}$$

$$= \frac{20}{27}$$

c

By symmetry, $\mathrm{E}(X) = 3$

4 **a**



b

$$\mathrm{f}'(x) = \frac{3}{64}(8x - 3x^2)$$

From the shape of the graph, the mode is at the point in $0 < x < 4$ at which $\mathrm{f}'(x) = 0$

$$x = \frac{8}{3}$$

c

Median m is such that $\int_{-\infty}^{m} \mathrm{f}(x)\ \mathrm{d}x = \frac{1}{2}$

$\int_0^m \frac{3}{64}(4x^2 - x^3)\ \mathrm{d}x = \left[\frac{4}{64}x^3 - \frac{3}{256}x^4\right]_0^m = \frac{16m^3 - 3m^4}{256} = \frac{1}{2}$

So $3m^4 - 16m^3 = -128$

$3m^4 - 16m^3 + 128 = 0$

5 **a**

$\mathrm{E}(Y) = \int_{-\infty}^{\infty} y\ \mathrm{f}(y)\ \mathrm{d}y$

$= \int_0^{\pi} \frac{y^2}{\pi} \sin y\ \mathrm{d}y$

$= 1.87$ (GDC)

$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} y^2\ \mathrm{f}(y)\ \mathrm{d}y$

$= \int_0^{\pi} \frac{y^3}{\pi} \sin y\ \mathrm{d}y$

$= 3.87$ (GDC)

$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2 = 0.379$

When you have studied Chapter 10, show that the exact values are

$\mathrm{E}(X) = \frac{\pi^2 - 4}{\pi}, \mathrm{E}(X^2) = \pi^2 - 6, \mathrm{Var}(X) = 2 - \frac{16}{\pi^2}$

b

$\mathrm{E}(4Y + 1) = 4\mathrm{E}(Y) + 1 = 8.47$

$\mathrm{Var}(4Y + 1) = 16\mathrm{Var}(Y) = 6.06$

6

$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{\mathrm{P}(A|B)\mathrm{P}(B)}{\mathrm{P}(A|B)\mathrm{P}(B) + \mathrm{P}(A|B')\mathrm{P}(B')} = \frac{0.6 \times 0.4}{0.6 \times 0.4 + 0.2 \times 0.6} = \frac{0.24}{0.36}$

$= \frac{2}{3}$

7 **a**

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{2}{26}$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{11}{26}$

b

$\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X = x) = \frac{80}{26} = \frac{40}{13}$

c

$\mathrm{E}(X^2) = \sum_x x^2\ \mathrm{P}(X = x) = \frac{135}{13}$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = \frac{135}{13} - \left(\frac{40}{13}\right)^2 = \frac{155}{169} \approx 0.917$

d

$\mathrm{Var}(20 - 5X) = 25Var(X) \approx 22.9 \approx 23$

Worked solutions

8 **a**

$H \sim \mathrm{B}\left(3, \frac{1}{2}\right)$

h	0	1	2	3
$\mathrm{P}(H = h)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b

$$\mathrm{E}(H) = \sum_h h\,\mathrm{P}(H = h) = \frac{3}{2}$$

$$\mathrm{E}(H^2) = \sum_h h^2\,\mathrm{P}(H = h) = 3$$

$$\mathrm{Var}(H) = \mathrm{E}(H^2) - \left(\mathrm{E}(H)\right)^2 = \frac{3}{4}$$

c

$W = 3H - 5$

$E(W) = \$(3E(H) - 5) = -\$0.50$

The game is not fair; in a fair game, the expected profit is zero.

d

$$\mathrm{Var}(3H - 5) = 9\mathrm{Var}(H) = \frac{27}{4} = 6.75 \text{ dollars}^2$$

9

$$\sum_x \mathrm{P}(X = x) = 1 = 0.5 + p + q$$

$p + q = 0.5(1)$

$$\mathrm{E}(X) = \sum_x x\,\mathrm{P}(X = x) = 1.1 + 4p + 5q = 3.3$$

$4p + 5q = 2.2(2)$

$(2) - 4(1): q = 0.2$

$p = 0.3, q = 0.2$

$$\mathrm{E}(X^2) = \sum_x x^2\,\mathrm{P}(X = x) = 12.5$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = 1.61$$

10 **a**

By symmetry, $E(Y) = 4$

b

$$\mathrm{E}(Y^2) = \sum_y y^2\,\mathrm{P}(Y = y) = 50a + 34b$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2 = 50a + 34b - 16 = 4.2$$

$50a + 34b = 20.2(1)$

$$\sum_y \mathrm{P}(Y = y) = 1 = 2a + 2b(2)$$

$(1) - 17(2): 16a = 3.2$

$a = 0.2, b = 0.3$

11

For a pdf $\mathrm{f}(x)$, $\int_{-\infty}^{\infty} \mathrm{f}(x)\ \mathrm{d}x = 1$

$$\int_1^3 ax + b\ \mathrm{d}x = \left[\frac{1}{2}ax^2 + bx\right]_1^3$$
$$= 4a + 2b = 1(1)$$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}(x)\ \mathrm{d}x$$

$$\int_1^3 ax^2 + bx\ \mathrm{d}x = \left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2\right]_1^3$$
$$= \frac{26}{3}a + 4b = 2.08(2)$$

$(2) - 2(1)$: $\frac{2}{3}a = 0.08$

$a = 0.12, b = 0.26$

12 **a**

$$\mathrm{P}(R) = \mathrm{P}(R|\text{Spain})\mathrm{P}(\text{Spain}) + \mathrm{P}(R|\text{Sweden})\mathrm{P}(\text{Sweden})$$
$$= 0.2 \times 0.3 + 0.6 \times 0.7$$
$$= 0.48$$

b

$$\mathrm{P}(\text{Sweden}|R) = \frac{\mathrm{P}(\text{Sweden} \cap R)}{\mathrm{P}(R)} = \frac{0.42}{0.48} = \frac{7}{8}$$

13 **a**

Let W be the event the canteen serves cheese sandwiches, and C be the event Emma eats in the canteen.

$\mathrm{P}(C|W) = 0.4, \mathrm{P}(C|W') = 0.7$

Let $\mathrm{P}(W) = p$

$\mathrm{P}(C) = \mathrm{P}(C|W)\mathrm{P}(W) + \mathrm{P}(C|W')\mathrm{P}(W') = 0.52$

$0.52 = 0.4p + 0.7(1 - \mathrm{p})$

$0.7 - 0.3p = 0.52$

$0.3p = 0.18$

$p = 0.6$

b

$$\mathrm{P}(W'|C) = \frac{\mathrm{P}(W' \cap C)}{\mathrm{P}(C)} = \frac{P(C|W') \times \mathrm{P}(W')}{\mathrm{P}(C)} = \frac{0.7(1-p)}{0.52} = \frac{0.28}{0.52} = \frac{7}{13} \approx 0.538$$

14

$\mathrm{P}(A \cap B_i) = \mathrm{P}(B_i) \times \mathrm{P}(A|B_i)$

	B_1	B_2	B_3	Total
A	0.12	0.24	0.1	0.46
A'	0.08	0.06	0.4	0.54
Total	0.2	0.3	0.5	1

$$\mathrm{P}(B_1|A) = \frac{\mathrm{P}(B_1 \cap A)}{\mathrm{P}(A)} = \frac{0.12}{0.46} = \frac{6}{23} \approx 0261$$

15 **a**

$$P(X > 2.5) = \int_{2.5}^{3} \frac{10}{81}(x^4 - 6x^3 + 9x^2)\ dx$$
$$= \left[\frac{1}{81}(2x^5 - 15x^4 + 30x^3)\right]_{2.5}^{3}$$
$$= 1 - \frac{1}{81}\left(2\left(\frac{625}{32}\right) - 15\left(\frac{625}{16}\right) + 30\left(\frac{125}{8}\right)\right)$$
$$= 1 - \frac{625}{648}$$
$$= \frac{23}{648} \approx 0.0355$$

b

$E(X) = 1.5$ by symmetry

$$E(X^2) = \int_{-\infty}^{\infty} x^2\ f(x)\ dx$$
$$= \int_{0}^{3} \frac{10}{81}(x^6 - 6x^5 + 9x^4)\ dx$$
$$\approx 2.57 \text{ (GDC)}$$

$\text{Var}(X) = E(X^2) - \big(E(X)\big)^2 \approx 0.321$

$\text{SD}(X) = \sqrt{\text{Var}(X)} \approx 0.567$

c

From part **a**, $P(X > 2.5) = 0.0355$

$80 \times 0.0355 = 2.84 \approx 3$

3 players should expect to win more than \$2.50.

d

The expected win is $E(X) = \$1.50$ so to be a fair game, the charge should also be \$1.50.

16

$E(X) = 200 \times \frac{1}{4} = 50$

$\text{Var}(X) = 200 \times \frac{1}{4} \times \frac{3}{4} = 37.5$

$\text{SD}(X) = \sqrt{\text{Var}(X)} = 6.12$

$P(43.88 < X < 56.12) = P(43 < X < 57)$ (since X can only take integer values)

$= 0.712$ (GDC)

So the probability that X will take a value more than one standard deviation from the mean is 0.288

17 **a**

By symmetry, $E(X) = 20$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\ f(x)\ dx$$
$$= \int_{10}^{30} \frac{3}{4000}(-x^4 + 40x^3 - 300x^2)\ dx$$
$$= 420 \text{ (GDC)}$$

$\text{Var}(X) = E(X^2) - \big(E(X)\big)^2 = 20$

b

$\mathrm{P}(X > 25) = \int_{25}^{30} \mathrm{f}(x)\ \mathrm{d}x = \frac{5}{32} = 0.15625$ (GDC)

c

If $Y \sim \mathrm{N}(20, 20)$ then $\mathrm{P}(Y > 25) = 0.132$

d

$\frac{41}{300} \approx 0.137$

This is closer to the value predicted by the student's model, suggesting that the normal distribution is more suitable for predicting.

18 **a**

$\int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = 1 = k \int_{0}^{6} x\mathrm{e}^{-x}\ \mathrm{d}x = 0.983k$

$k = \frac{1}{0.983} \approx 1.018$

Use techniques in Chapter 10 to show that k is exactly given by $k = \frac{1}{1 - 7\mathrm{e}^{-6}}$

b

$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\ \mathrm{f}(x)\ \mathrm{d}x$

$= k \int_{0}^{6} x^2\mathrm{e}^{-x}\ \mathrm{d}x = 1.91$

Mean lifetime is 1.91 years $\approx$ 23 months

c

$\mathrm{P}(X < 1) = \int_{0}^{1} \mathrm{f}(x)\ \mathrm{d}x \approx 0.269$

d

$\mathrm{P}(X < 0.5) = 0.0918$

$\mathrm{P}(X < 2) = 0.604$

di

Let A be the number from five which fail in less than six months.

$A \sim \mathrm{B}(5, 0.0918)$

$\mathrm{P}(A = 0) = (1 - 0.0918)^5 = 0.618$

dii

Let C be the number from five which fail in less than two years.

$C \sim \mathrm{B}(5, 0.604)$

$\mathrm{P}(C < 5) = 1 - \mathrm{P}(C = 5)$

$= 1 - 0.604^5$

≈ 0.919

19

After a blue ball is drawn, there are 6 blue balls remaining out of a total of $6 + 5 + 8 = 19$

$\mathrm{P}(B_2|B_1) = \frac{6}{19}$

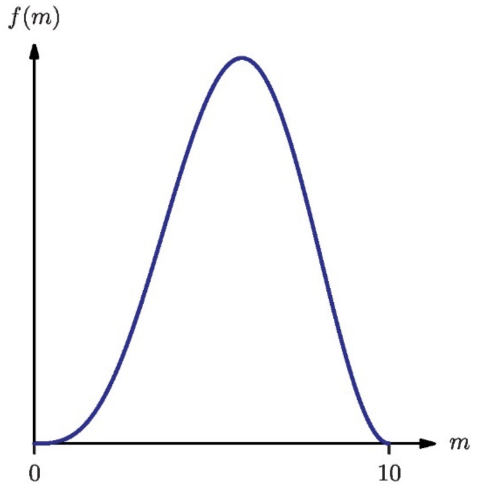
20 **a**

$$\int_{-\infty}^{\infty} \mathrm{f}(m)\,\mathrm{d}m = 1 = k\int_0^{10} m\sin^2\left(\frac{\pi m}{10}\right)\mathrm{d}m = 25k \text{ (GDC)}$$

Use techniques from Chapter 10 to show that this is the value of the integral

$$k = \frac{1}{25}$$

b



c

$$\mathrm{P}(M > 6) = \int_6^{10} \mathrm{f}(m)\ \mathrm{d}m = 0.436 \text{ (GDC)}$$

d

$$\mathrm{E}(M) = \int_{-\infty}^{\infty} m\,\mathrm{f}(m)\ \mathrm{d}m = 5.65 \text{ kg (GDC)}$$

$$\mathrm{E}(M^2) = \int_{-\infty}^{\infty} m^2\,\mathrm{f}(m)\ \mathrm{d}m = 34.802 \text{ kg}^2 \text{(GDC)}$$

$$\mathrm{Var}(M) = \mathrm{E}(M^2) - \left(\mathrm{E}(M)\right)^2 \approx 2.84 \text{ kg}^2$$

$$\mathrm{SD}(M) = \sqrt{\mathrm{Var}(M)} \approx 1.69 \text{ kg}$$

e

Let $Q \sim \mathrm{N}(5.65, 2.84)$

$\mathrm{P}(Q > 6) = 0.417$

$$\text{Percentage error} = \frac{|\text{True value} - \text{estimated value}|}{|\text{true value}|} \times 100\%$$

$$\approx 4\%$$

21

Let D be the event that a fly dies within the first three days and M be the event that it has the mutation.

$\mathrm{P}(D|M') = 0.1, \mathrm{P}(D|M) = 0.9, \mathrm{P}(M) = 0.03$

	M	M'	Total
D	0.027	0.097	0.124
D'	0.003	0.873	0.876
Total	0.03	0.97	1

$$\mathrm{P}(M'|D) = \frac{0.097}{0.124} \approx 0.782$$

22

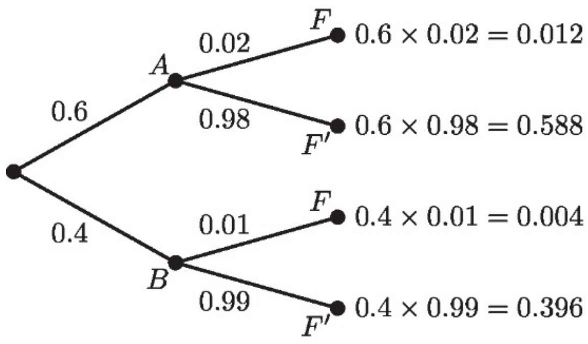
Let L be the event that the bus is late and R the event that it rains.

$\mathrm{P}(R) = \frac{9}{20}, \mathrm{P}(L|R') = \frac{3}{20}, \mathrm{P}(L|R) = \frac{7}{20}$

	R	R'	Total
L	0.1575	0.0825	0.24
L'	0.2925	0.4675	0.76
Total	0.45	0.55	1

$\mathrm{P}(R'|L) = \frac{0.0825}{0.24} = 0.34375 = \frac{11}{32}$

23 **ai**



aii

$\mathrm{P}(F) = 0.012 + 0.004 = 0.016$

aiii

$\mathrm{P}(A|F) = \frac{\mathrm{P}(A \cap F)}{\mathrm{P}(F)} = \frac{0.012}{0.016} = \frac{3}{4}$

bi

If F_i is the event that the i^{th} transistor selected is faulty.

$$\mathrm{P}(X = 2) = \mathrm{P}(F_1F_2F_3') + \mathrm{P}(F_1F_2'F_3) + \mathrm{P}(F_1'F_2F_3)$$
$$= \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} + \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}$$
$$= \frac{12}{35}$$

bii

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

biii

$$\mathrm{E}(X) = \sum_x x\,\mathrm{P}(X = x) = \frac{1}{35}(0 \times 4 + 1 \times 18 + 2 \times 12 + 3 \times 1) = \frac{45}{35} = \frac{9}{7}$$

24 **a**

$$\mathrm{E}(T) = \int_{-\infty}^{\infty} t\,\mathrm{f}(t)\ \mathrm{d}t$$
$$= \int_0^2 \frac{1}{12}(8t^2 - t^4)\ \mathrm{d}t$$
$$= \left[\frac{1}{180}(40t^3 - 3t^5)\right]_0^2$$
$$= \frac{320 - 96}{180}$$
$$= \frac{56}{45}$$

b

Since the function has a local maximum within the domain as its only turning point, this must be the mode.

$$\mathrm{f}'(t) = \frac{1}{12}(8 - 3t^2)$$

$$\mathrm{f}'(t) = 0 \text{ when } t = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}$$

c

Median m is such that $\int_{-\infty}^{m} \mathrm{f}(t)\ \mathrm{d}t = \frac{1}{2}$

$$\int_0^m \frac{1}{12}(8t - t^3)\ \mathrm{d}t = \left[\frac{1}{48}(16t^2 - t^4)\right]_0^m = \frac{1}{48}(16m^2 - m^4) = \frac{1}{2}$$

$m^4 - 16m^2 + 24 = 0$

$m^2 = 8 \pm \sqrt{40}$

Selecting negative root since $m < 2$:

$m^2 = 8 - 2\sqrt{10}$

Hence $m = \sqrt{8 - 2\sqrt{10}}$

25 **ai**

$$\mathrm{P}(T > 5) = \int_5^6 \mathrm{f}(t)\ \mathrm{d}t = 0.407 \text{ (GDC)}$$

aii

$$\mathrm{P}(5 < T < 5.5) = \int_5^{5.5} \mathrm{f}(t)\ \mathrm{d}t = 0.275 \text{ (GDC)}$$

b

Let X be the number of batteries, out of three, that last at least 500 hours

$X \sim \mathrm{B}(3, 0.407)$

$\mathrm{P}(X = 3) = 0.407^3 = 0.0676$

c

Let Y be the number of batteries, out of three, that last at least 550 hours

$Y \sim \mathrm{B}(3, 0.132)$

$\mathrm{P}(Y = 3) = 0.132^3 = 0.00230$

$$\mathrm{P}(Y = 3 | X = 3) = \frac{\mathrm{P}(Y = 3 \cap X = 3)}{\mathrm{P}(X = 3)} = \frac{\mathrm{P}(Y = 3)}{\mathrm{P}(X = 3)} = \frac{0.00230}{0.0676} = 0.0340$$

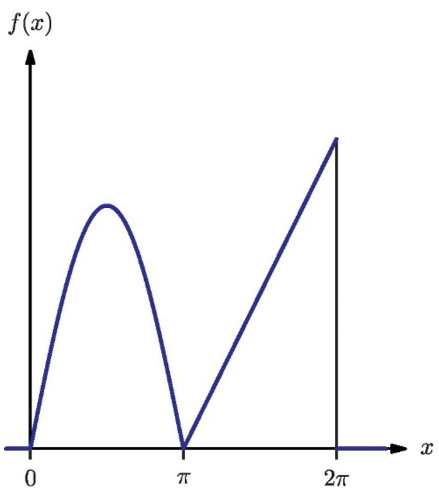
26
Let X be the mass of the chicken and A, B, C the event of the chicken being the indicated breed.
$\mathrm{P}(X > 1.8|A) = 0.159$
$\mathrm{P}(X > 1.8|B) = 0.00135$
$\mathrm{P}(X > 1.8|C) = 0.579$
Then $\mathrm{P}(X > 1.8) = 0.2(0.159) + 0.45(0.00135) + 0.35(0.579) = 0.235$
$$\mathrm{P}(B|X > 1.8) = \frac{\mathrm{P}(B \cap X > 1.8)}{\mathrm{P}(X > 1.8)} = \frac{0.45(0.00135)}{0.235} = 0.00258$$

27 **a**
$$\mathrm{P}(X < 1) = \int_0^1 \mathrm{f}(x)\ \mathrm{d}x = [-e^{-ax}]_0^1 = 1 - e^{-a} = 1 - \frac{1}{\sqrt{2}}$$
$$\mathrm{e}^{-a} = \frac{1}{\sqrt{2}}$$
$$a = -\ln\left(\frac{1}{\sqrt{2}}\right) = \ln(\sqrt{2}) = \frac{1}{2}\ln 2$$

b
Median m is such that $\int_0^m \mathrm{f}(x)\ \mathrm{d}x = \frac{1}{2}$
$$\int_0^m a\mathrm{e}^{-ax}\ \mathrm{d}x = [-\mathrm{e}^{-ax}]_0^m = 1 - \mathrm{e}^{-am} = \frac{1}{2}$$
$$\mathrm{e}^{-am} = \frac{1}{2} = (\mathrm{e}^{-a})^m = \left(\frac{1}{\sqrt{2}}\right)^m$$
$$m = 2$$

c
$$\mathrm{P}(X < 3) = \int_0^3 \mathrm{f}(x)\ \mathrm{d}x = 1 - \mathrm{e}^{-3a} = 1 - \frac{\sqrt{2}}{4}$$
$$\mathrm{P}(X < 3|X > 1) = \frac{\mathrm{P}(1 < X < 3)}{\mathrm{P}(X > 1)} = \frac{\left(1 - \frac{\sqrt{2}}{4}\right) - \left(1 - \frac{\sqrt{2}}{2}\right)}{\frac{1}{\sqrt{2}}} = \frac{\left(\frac{\sqrt{2}}{4}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{2}$$

28 **a**



b

$$P(X \leq \pi) = \int_0^{\pi} f(x)\ dx$$
$$= \left[-\frac{1}{4}\cos x\right]_0^{\pi}$$
$$= \frac{1}{2}$$

c

For the $f(x)$ *to* be a pdf, require $\int_{-\infty}^{\infty} f(x)\ dx = 1$

So $\int_{\pi}^{2\pi} f(x)\ dx = \frac{1}{2}$

$$\int_{\pi}^{2\pi} a(x-\pi)\ dx = \left[\frac{a}{2}(x^2 - 2\pi x)\right]_{\pi}^{2\pi} = \frac{a}{2}\left(4\pi^2 - 4\pi^2 - (\pi^2 - 2\pi^2)\right) = \frac{\pi^2 a}{2}$$
$$= \frac{1}{2}$$

$$a = \frac{1}{\pi^2}$$

d

Median m is such that $\int_{-\infty}^{m} f(x)\ dx = \frac{1}{2}$

From part **a**, the median is π.

e

$$E(X) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$
$$= \int_0^{\pi} \frac{x}{4}\sin x\ dx + \int_{\pi}^{2\pi} \frac{x^2 - \pi x}{\pi^2}\ dx$$
$$= 3.40\ (\text{GDC})$$

f

$$E(X^2) = \int_{-\infty}^{\infty} x^2\ f(x)\ dx$$
$$= \int_0^{\pi} \frac{x^2}{4}\sin x\ dx + \int_{\pi}^{2\pi} \frac{x^3 - \pi x^2}{\pi^2}\ dx$$
$$= 15.45(\text{GDC})$$

$$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 3.87$$

Use methods from Chapter 10 to show that the exact values are

$E(X) = \frac{13\pi}{12}, E(X^2) = \frac{5\pi^2}{3} - 1, \text{Var}(X) = \frac{71\pi^2}{144} - 1$

Worked solutions

g

$\mathrm{P}\left(\frac{\pi}{2} < X < \pi\right) = \frac{1}{4}$ (by symmetry; the left half of the distribution has area $\frac{1}{2}$)

$\mathrm{P}\left(\pi < X < \frac{3\pi}{2}\right)$

$= \frac{1}{8}$ (whole triangle has area $\frac{1}{2}$ and this triangle has half the width and height)

Then $\mathrm{P}\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right) = \frac{3}{8}$

h

$$\mathrm{P}\left(\pi < X < 2\pi \middle| \frac{\pi}{2} < X < \frac{3\pi}{2}\right) = \frac{\mathrm{P}\left(\pi < X < 2\pi \middle| \frac{\pi}{2} < X < \frac{3\pi}{2}\right)}{\mathrm{P}\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right)}$$

$$= \frac{\mathrm{P}\left(\pi < X < \frac{3\pi}{2}\right)}{\mathrm{P}\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right)}$$

$$= \frac{\left(\frac{1}{8}\right)}{\left(\frac{3}{8}\right)} \text{ by part } \mathbf{g}$$

$$= \frac{1}{3}$$

Worked solutions

10 Further calculus

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 10A

Note: Throughout these worked solutions, we shall use, for clarity, braces around expressions in limits. This has not been standard notation in the textbook and is not required from students, but assists in keeping clear what is and is not within an individual limit.

31

$\mathrm{f}(x) = \mathrm{e}^{5x}$

Proposition: $\mathrm{f}^n(x) = 5^n \mathrm{e}^{5x}$ for $n \geq 0$

Base case: $\mathrm{f}^0(x) = \mathrm{f}(x) = 5^0 \mathrm{e}^{5x} = \mathrm{e}^{5x}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $\mathrm{f}^k(x) = 5^k \mathrm{e}^{5x}$

Working towards: $\mathrm{f}^{k+1}(x) = 5^{x+1}\mathrm{e}^{5x}$

$$\begin{aligned} \mathrm{f}^{k+1}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}^k(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(5^k \mathrm{e}^{5x}) \text{ using the assumption} \\ &= 5^k \times 5\mathrm{e}^{5x} \\ &= 5^{k+1}\mathrm{e}^{5x} \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

32 **ai**

$$\mathrm{f}(x) = \frac{3}{\mathrm{e}^{-x} + 2}$$

$$\lim_{x\to\infty}\{\mathrm{f}(x)\} = \frac{3}{2}$$

aii

$$\mathrm{f}(x) = \frac{3\mathrm{e}^x}{1 + 2\mathrm{e}^x}$$

$$\lim_{x\to-\infty}\{\mathrm{f}(x)\} = \frac{0}{1+0} = 0$$

aiii

$$\mathrm{f}(x) = \frac{3\mathrm{e}^x}{1 + 2\mathrm{e}^x}$$

$$\lim_{x\to 0}\{\mathrm{f}(x)\} = \frac{3}{1+2} = 1$$

b

Let $u = 3e^x, v = 1 + 2e^x$ so $u' = 3e^x, v' = 2e^x$

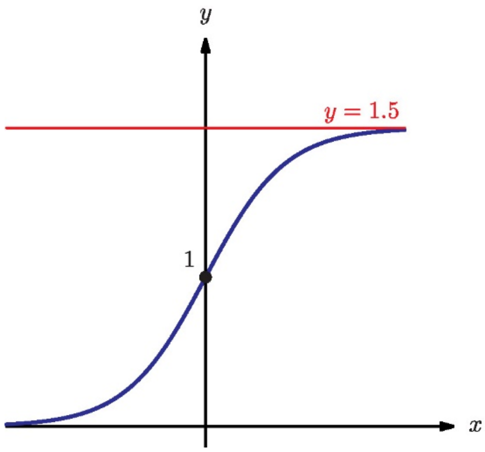
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

$$f'(x) = \frac{3e^x(1 + 2e^x) - 6e^{2x}}{(1 + 2e^x)^2} = \frac{3e^x}{(1 + 2e^x)^2}$$

Since both numerator and denominator are always greater than zero, $f'(x) > 0$ for all x.

The function $f(x)$ is therefore always increasing.

c



33

$$f(x) = \begin{cases} e^{ax} + x & x \le 2 \\ 2e^{ax} & x > 2 \end{cases}$$

$f(2) = e^{2a} + 2 = 2e^{2a}$

$e^{2a} = 2$

$a = \ln\sqrt{2}$

34 **a**

$$f(x) = a^x + \frac{3x}{2}$$

As $x \to -\infty$, a^x converges to zero but $\frac{3x}{2}$ diverges, so the sum must diverge.

b

$$f(x) = \begin{cases} a^x + \frac{3x}{2} & x \le 2 \\ 4a^{x-1} & x > 2 \end{cases}$$

$f(2) = a^2 + 3 = 4a$

$a^2 - 4a + 3 = 0$

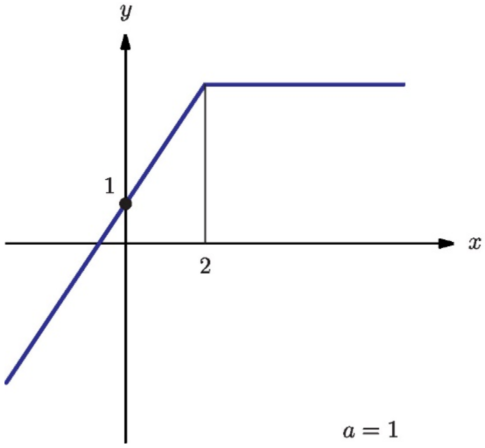
$(a - 1)(a - 3) = 0$

$a = 1$ or $a = 3$

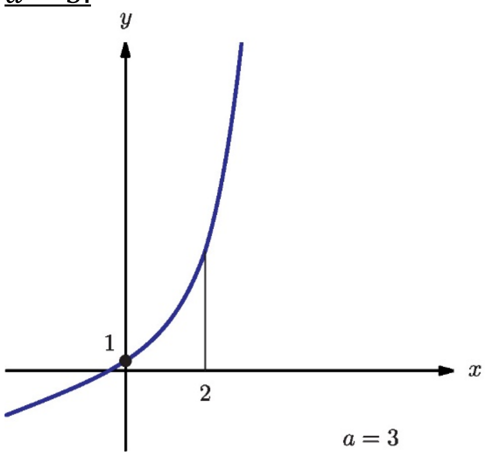
Worked solutions

c

<u>$a = 1$:</u>



<u>$a = 3$:</u>



35

$\mathrm{f}(x) = \begin{cases} ax^2 + x & x \leq 2 \\ bx + 2 & x > 2 \end{cases}$

$\mathrm{f}'(x) = \begin{cases} 2ax + 1 & x \leq 2 \\ b & x > 2 \end{cases}$

$\mathrm{f}(2) = 4a + 2 = 2b + 2 \Rightarrow 4a = 2b(1)$

$\mathrm{f}'(2) = 4a + 1 = b(2)$

Substituting (1) into (2): $2b + 1 = b$

$b = -1, a = -0.5$

36

$y = xe^{2x}$

<u>Proposition:</u> $y^{(n)}(x) = (n2^{n-1} + 2^n x)e^{2x}$ for $n \geq 0$

<u>Base case:</u> $y^{(0)}(x) = y = xe^{2x} = (0 \times 2^{-1} + 2^0 x)e^{2x}$ so the proposition is true for $n = 0$

<u>Inductive step:</u> Assume the proposition is true for $n = k \geq 0$

So $y^{(k)}(x) = (k2^{k-1} + 2^k x)e^{2x}$

<u>Working towards:</u> $y^{(k+1)}(x) = \left((k+1)2^k + 2^{k+1}x\right)e^{2x}$

$$\begin{aligned} y^{(k+1)}(x) &= \frac{d}{dx}f^k(x) \\ &= \frac{d}{dx}\left((k2^{k-1} + 2^k x)e^{2x}\right) \text{ using the assumption} \\ &= 2(k2^{k-1} + 2^k x)e^{2x} + 2^k e^{2x} \\ &= \left((k+1)2^k + 2^{k+1}x\right)e^{2x} \end{aligned}$$

So the proposition is true for $n = k + 1$

<u>Conclusion</u>:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

37 **a**

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \lim_{h\to 0}\left\{\frac{\left(\frac{1}{1-x-h}\right) - \left(\frac{1}{1-x}\right)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{\left(\frac{1-x-(1-x-h)}{(1-x-h)(1-x)}\right)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{\left(\frac{h}{(1-x-h)(1-x)}\right)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{1}{(1-x-h)(1-x)}\right\} \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

b

$f(x) = \frac{1}{1-x}$

<u>Proposition:</u> $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ for $n \geq 0$

<u>Base case:</u> $f^{(0)}(x) = f(x) = \frac{1}{1-x} = \frac{0!}{(1-x)^{0+1}}$ so the proposition is true for $n = 0$

<u>Inductive step:</u> Assume the proposition is true for $n = k \geq 0$

So $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$

<u>Working towards:</u> $f^{(k+1)}(x) = \frac{(k+1)!}{(1-x)^{k+2}}$

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} f^k(x) \\
&= \frac{d}{dx}\left(\frac{k!}{(1-x)^{k+1}}\right) \text{ using the assumption} \\
&= \frac{d}{dx}(k!\,(1-x)^{-k-1}) \\
&= -k!\,(-k-1)(1-x)^{-k-2} \text{ using chain rule} \\
&= \frac{(k+1)!}{(1-x)^{k+2}}
\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

38

$f(x) = \sin x$

The proof will use the fact that

$$\begin{aligned}
\sin\left(A + \frac{\pi}{2}\right) &= \sin A \cos\left(\frac{\pi}{2}\right) + \cos A \sin\left(\frac{\pi}{2}\right) \\
&= \cos A
\end{aligned}$$

So $\sin\left(A + \frac{\pi}{2}\right) = \cos A$ for all $A (*)$

Proposition: $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

Base case: $f^0(x) = f(x) = \sin x = \sin\left(x + \frac{0\pi}{2}\right)$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $f^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$

Working towards: $f^{(k+1)}(x) = \sin\left(x + \frac{(k+1)\pi}{2}\right)$

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} f^k(x) \\
&= \frac{d}{dx}\left(\sin\left(x + \frac{k\pi}{2}\right)\right) \text{ using the assumption} \\
&= \cos\left(x + \frac{k\pi}{2}\right) \\
&= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right) \text{ using } (*) \\
&= \sin\left(x + \frac{(k+1)\pi}{2}\right)
\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

39

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \lim_{h\to 0}\left\{\frac{\sqrt{x+h}-\sqrt{x}}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{1}{\sqrt{x+h}+\sqrt{x}}\right\}$$
$$= \frac{1}{2\sqrt{x}}$$

40

Tip: Remember that you should never use the same letter for two meanings in any working. Since h is used in the question as a function, we need to use a different letter for the infinitessimal in the limit. Greek letters delta (δ) and epsilon (ϵ) are often used in this context in mathematical analysis.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{g}(x)+\mathrm{h}(x)\right) = \lim_{\delta\to 0}\left\{\frac{\left(\mathrm{g}(x+\delta)+\mathrm{h}(x+\delta)\right)-\left(\mathrm{g}(x)+\mathrm{h}(x)\right)}{\delta}\right\}$$
$$= \lim_{\delta\to 0}\left\{\frac{\mathrm{f}(x+\delta)-\mathrm{f}(x)}{\delta}+\frac{\mathrm{g}(x+\delta)-\mathrm{g}(x)}{\delta}\right\}$$
$$= \lim_{\delta\to 0}\left\{\frac{\mathrm{f}(x+\delta)-\mathrm{f}(x)}{\delta}\right\}+\lim_{h\to 0}\left\{\frac{\mathrm{g}(x+\delta)-\mathrm{g}(x)}{\delta}\right\}$$
$$= \mathrm{f}'(x)+\mathrm{g}'(x)$$

This assumes that $\mathrm{f}'(x)$ and $\mathrm{g}'(x)$ are well-defined.

There is some concern about rigour here – it is not necessarily true that the limit of a sum always equals the sum of limits (for example, the sum of x and $-x$ as $x \to \infty$ clearly has limit zero, but the individual limits are not defined). However, it is the case that if both expressions have a finite limit (as here) then it is legitimate to split the limit of the sum of two expressions into the sum of limits.

41

$\mathrm{f}(x) = x\sin x$

<u>Proposition:</u> $\mathrm{f}^{(2n)}(x) = (-1)^n(x\sin x - 2n\cos x)$

<u>Base case:</u>

$\mathrm{f}^0(x) = \mathrm{f}(x) = x\sin x = (-1)^0(x\sin x - 2\times 0\cos x)$so the proposition is true for $n = 0$

<u>Inductive step:</u> Assume the proposition is true for $n = k \geq 0$

So $\mathrm{f}^{(2k)}(x) = (-1)^k(x\sin x - 2k\cos x)$

<u>Working towards:</u> $\mathrm{f}^{(2k+2)}(x) = (-1)^{k+1}(x\sin x - 2(k+1)\cos x)$

$$\begin{aligned}
f^{(2k+2)}(x) &= \frac{d}{dx}\left(\frac{d}{dx}f^k(x)\right) \\
&= \frac{d}{dx}\left(\frac{d}{dx}((-1)^k(x\sin x - 2k\cos x)\right) \text{ using the assumption} \\
&= \frac{d}{dx}\big((-1)^k(\sin x + x\cos x + 2k\sin x)\big) \\
&= (-1)^k(\cos x + \cos x - x\sin x + 2k\cos x) \\
&= (-1)^k\big((2k+2)\cos x - x\sin x\big) \\
&= (-1)^{k+1}(x\sin x - (2k+2)\cos x)
\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

42 $\quad f(x) = \ln x$

Proposition: $\quad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$

Base case:

$f^1(x) = f'(x) = \frac{1}{x} = \frac{(-1)^2 1!}{x^1}$ so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $f^{(k)}(x) = \dfrac{(-1)^{k+1}(k-1)!}{x^k}$

Working towards: $f^{(k+1)}(x) = \frac{(-1)^{k+2}k!}{x^{k+1}}$

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx}\big(f^k(x)\big) \\
&= \frac{d}{dx}((-1)^{k+1}(k-1)!\,x^{-k}) \text{ using the assumption} \\
&= (-1)^{k+1}(k-1)!\,\big(-kx^{-(k+1)}\big) \\
&= (-1)^{k+2}k!\,x^{-(k+1)} \\
&= \frac{(-1)^{k+2}k!}{x^{k+1}}
\end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

43

$\big(\sqrt{x+4} - \sqrt{x}\big)\big(\sqrt{x+4} + \sqrt{x}\big) = x + 4 - x = 4$

So $\big(\sqrt{x+4} - \sqrt{x}\big) = \dfrac{4}{\sqrt{x+4} + \sqrt{x}}$

Then $\lim\limits_{x\to\infty}\{\sqrt{x+4} - \sqrt{x}\} = \lim\limits_{x\to\infty}\left\{\dfrac{4}{\sqrt{x+4} + \sqrt{x}}\right\}$

The numerator within the limit is constant and the denominator tends to infinity as $x \to \infty$

Hence the limit of this ratio is zero.

Exercise 10B

9

Using L'Hôpital's rule:

$$\lim_{x\to 0}\left\{\frac{e^x - 1}{x}\right\} = \lim_{x\to 0}\left\{\frac{e^x}{1}\right\} = 1$$

10

Using L'Hôpital's rule:

$$\lim_{x\to \infty}\left\{\frac{x^2}{e^x}\right\} = \lim_{x\to \infty}\left\{\frac{2x}{e^x}\right\} = \lim_{x\to \infty}\left\{\frac{2}{e^x}\right\} = 0$$

11

Using L'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 0}\left\{\frac{1-\cos(x^2)}{x^4}\right\} &= \lim_{x\to 0}\left\{\frac{2x\sin(x^2)}{4x^3}\right\} \quad \text{using L'Hôpital's rule}\\ &= \lim_{x\to 0}\left\{\frac{\sin(x^2)}{2x^2}\right\} \text{(cancellation)}\\ &= \lim_{x\to 0}\left\{\frac{2x\cos(x^2)}{4x}\right\} \text{ using L'Hôpital's rule}\\ &= \lim_{x\to 0}\left\{\frac{\cos(x^2)}{2}\right\} \text{ (cancellation)}\\ &= \frac{1}{2}\end{aligned}$$

12 **a**

$$\lim_{x\to 0}\left\{\frac{x-\cos x}{x+\cos x}\right\} = -1$$

(The limit of the numerator is -1 and the limit of the denominator is 1 so the limit of the ratio can be determined directly)

b

Using L'Hôpital's rule:

$$\lim_{x\to 0}\left\{\frac{x-\sin x}{x+\sin x}\right\} = \lim_{x\to 0}\left\{\frac{1-\cos x}{1+\cos x}\right\} = 0$$

13 **a**

Using L'Hôpital's rule:

$$\lim_{x\to 0}\left\{\frac{\sin x}{x^2}\right\} = \lim_{x\to 0}\left\{\frac{\cos x}{2x}\right\} = \infty$$

b

Using L'Hôpital's rule:

$$\lim_{x\to 2}\left\{\frac{\sin(x-2)}{x^2-4}\right\} = \lim_{x\to 2}\left\{\frac{\cos(x-2)}{2x}\right\} = \frac{1}{4}$$

14

$$\lim_{x\to\frac{\pi}{2}}\left\{\frac{\cos^2(5x)}{\cos^2 x}\right\} = \lim_{x\to\frac{\pi}{2}}\left\{\frac{-10\sin(5x)\cos(5x)}{-2\sin x\cos x}\right\} \quad \text{using L'Hôpital's rule}$$
$$= \lim_{x\to\frac{\pi}{2}}\left\{\frac{5\sin(10x)}{\sin(2x)}\right\} \quad \text{using double angle formulae}$$
$$= \lim_{x\to\frac{\pi}{2}}\left\{\frac{50\cos(10x)}{2\cos(2x)}\right\} \quad \text{using L'Hôpital's rule}$$
$$= 25$$

15

$$\lim_{x\to 1}\left\{\frac{(\ln x)^2}{x^2-2x+1}\right\} = \lim_{x\to 1}\left\{\frac{2x^{-1}\ln x}{2x-2}\right\} \quad \text{using L'Hôpital's rule}$$
$$= \lim_{x\to 1}\left\{\frac{\ln x}{x^2-x}\right\} \quad \text{simplifying}$$
$$= \lim_{x\to 1}\left\{\frac{x^{-1}}{2x-1}\right\} \quad \text{using L'Hôpital's rule}$$
$$= \lim_{x\to 1}\left\{\frac{1}{2x^2-x}\right\} \quad \text{simplifying}$$
$$= 1$$

16

$$\lim_{x\to 1}\left\{\frac{(\ln x)^2}{x^3+x^2-5x+3}\right\} = \lim_{x\to 1}\left\{\frac{2x^{-1}\ln x}{3x^2+2x-5}\right\} \quad \text{using L'Hôpital's rule}$$
$$= \lim_{x\to 1}\left\{\frac{2\ln x}{3x^3+2x^2-5x}\right\} \quad \text{simplifying}$$
$$= \lim_{x\to 1}\left\{\frac{2x^{-1}}{9x^2+4x-5}\right\} \quad \text{using L'Hôpital's rule}$$
$$= \lim_{x\to 1}\left\{\frac{2}{9x^3+4x^2-5x}\right\} \quad \text{simplifying}$$
$$= \frac{1}{4}$$

17 **a**

$$\lim_{x\to 0}\left\{\frac{\sin x}{x}\right\} = \lim_{x\to 0}\left\{\frac{\cos x}{1}\right\} \quad \text{using L'Hôpital's rule}$$
$$= 1$$

b

Considering that $\frac{1}{|x|} \le \left|\frac{\sin x}{x}\right| \le \frac{1}{|x|}$ for all x:

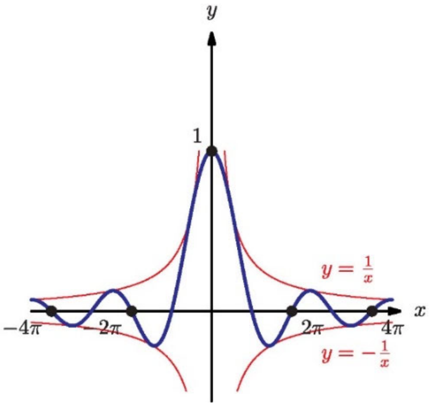
$$-\lim_{x\to\infty}\left\{\frac{1}{x}\right\} \le \lim_{x\to\infty}\left\{\frac{\sin x}{x}\right\} \le \lim_{x\to\infty}\left\{\frac{1}{x}\right\}$$

$$0 \le \lim_{x\to\infty}\left\{\frac{\sin x}{x}\right\} \le 0$$

Hence $\lim_{x\to\infty}\left\{\frac{\sin x}{x}\right\} = 0$

Tip: The limit of $\sin x$ as $x \to \infty$ is neither zero nor infinity; for a sinusoidal the limit is simply not defined, but the function does exist for a finite interval range; in such a situation, this sort of trapping inequality can be the simplest approach to finding limits at infinity.

c



18

Using L'Hôpital's rule,

$$\lim_{x\to 0}\left\{\frac{\sin 3x}{\sin x}\right\} = \lim_{x\to 0}\left\{\frac{3\cos 3x}{\cos x}\right\} = 3$$

$$\lim_{x\to 0}\left\{\frac{e^{3x}-1}{x}\right\} = \lim_{x\to 0}\left\{\frac{3e^{3x}}{1}\right\} = 3$$

Hence the piecewise function is continuous at 3.

19

Using L'Hôpital's rule,

$$\lim_{x\to -\infty}\{xe^x\} = \lim_{x\to -\infty}\left\{\frac{x}{e^{-x}}\right\} = \lim_{x\to -\infty}\left\{\frac{1}{-e^{-x}}\right\} = -\lim_{x\to -\infty}\{e^x\} = 0$$

20

$$\begin{aligned}\lim_{x\to 0}\left\{\frac{1}{\sin x}-\frac{1}{x}\right\} &= \lim_{x\to 0}\left\{\frac{x-\sin x}{x\sin x}\right\}\\ &= \lim_{x\to 0}\left\{\frac{1-\cos x}{x\cos x+\sin x}\right\} \text{ using L'Hôpital's rule}\\ &= \lim_{x\to 0}\left\{\frac{\sin x}{2\cos x - x\sin x}\right\} \text{ using L'Hôpital's rule}\\ &= 0\end{aligned}$$

21

$$\begin{aligned}\lim_{x\to 0}\left\{\frac{1}{x}-\frac{1}{e^x-1}\right\} &= \lim_{x\to 0}\left\{\frac{e^x-1-x}{x(e^x-1)}\right\}\\ &= \lim_{x\to 0}\left\{\frac{e^x-1}{e^x+xe^x-1}\right\} \text{ using L'Hôpital's rule}\\ &= \lim_{x\to 0}\left\{\frac{e^x}{2e^x+xe^x}\right\} \text{ using L'Hôpital's rule}\\ &= \frac{1}{2}\end{aligned}$$

22

$$\begin{aligned}\lim_{x\to \infty}\left\{\frac{e^x+e^{-x}}{e^x-e^{-x}}\right\} &= \lim_{x\to \infty}\left\{\frac{e^{2x}+1}{e^{2x}-1}\right\}\\ &= \lim_{x\to \infty}\left\{\frac{2e^{2x}}{2e^{2x}}\right\} \text{ using L'Hôpital's rule}\\ &= 1\end{aligned}$$

This limit can be resolved without using L'Hôpital's rule by an alternative manipulation and then direct application of limits:

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to\infty} \left\{\frac{1 + e^{-2x}}{1 - e^{-2x}}\right\} = 1$$

However, since the question specified method this is one of the rare cases where the simpler method is not appropriate, even though it is valid and rigorous!

Exercise 10C

10 **a**

When $x = 1, y = 2, 3x^2 + y^3 = 3(1)^2 + 2^3 = 3 + 8 = 11$

So $(1, 2)$ does lie on the curve.

b

Implicit differentiation:

$$6x + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -2xy^{-2}$$

Gradient at $(1, 2)$ is $-\frac{2}{4} = -\frac{1}{2}$

c

Perpendicular gradient is 2

Normal has equation $y - 2 = 2(x - 1)$

$y = 2x$

11 **a**

When $x = 0$, $\ln y = 0$ so $y = 1$

y-intercept is $(0, 1)$

b

Implicit differentiation:

$$\frac{1}{y}\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = y\cos x$$

At $(0, 1)$, gradient is 1

Tangent has equation $y = x + 1$

12 **a**

$e^x + \ln y = 0$

When $x = a, y = e^{-0.5}$

Substituting: $e^a = 0.5$

$a = \ln 0.5 = -\ln 2$

b

Implicit differentiation:

$$e^x + y^{-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -ye^x$$

At $A(\ln 0.5\,, e^{-0.5})$, gradient is $0.5e^{-0.5} = -\frac{1}{2\sqrt{e}}$

13

Implicit differentiation:

$$2x - 2 - (3y^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-2}{3y^2-1}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(3,1)} = \frac{4}{2} = 2$$

Equation of the tangent at $(3, 1)$ is $y - 1 = 2(x - 3)$

$y = 2x - 5$

14

$2xy - 2y^2 = x + y$

Implicit differentiation:

$$2y + (2x - 4y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-2y}{2x-4y-1}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(3,1)} = \frac{-1}{1} = -1$$

Tangent at $(3, 1)$ has equation $y - 1 = -(x - 3)$

$y = 4 - x$

15 $\sin(x + y) = \sqrt{2}\cos(x - y)$

a

When $x = \frac{13\pi}{24}$ and $y = \frac{5\pi}{24}$

$\sin(x + y) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sqrt{2}\cos(x - y) = \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{\sqrt{2}}{2}$

So $\sin(x + y) = \sqrt{2}\cos(x - y)$ at $\left(\frac{13\pi}{24}, \frac{5\pi}{24}\right)$

Therefore this point does lie on the curve.

b

Implicit differentiation and chain rule:

$$\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos(x + y) = -\sqrt{2}\left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)\sin(x - y)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos(x + y) + \sqrt{2}\sin(x - y)}{\sqrt{2}\sin(x - y) - \cos(x + y)}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\left(\frac{13\pi}{24},\frac{5\pi}{24}\right)} = \frac{-\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}}{\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}}$$

$$= \frac{\left(\sqrt{6} - \sqrt{2}\right)^2}{\left(\sqrt{6} + \sqrt{2}\right)\left(\sqrt{6} - \sqrt{2}\right)}$$

$$= \frac{\left(\sqrt{6} - \sqrt{2}\right)^2}{6 - 2}$$

$$= \frac{6 + 2 - 2\sqrt{12}}{4}$$

$$= 2 - \sqrt{3}$$

16

$x^2 + 3xy + y^2 = 1$ so when $x = 0$, $y^2 = 1$.
The two y-intercepts of the curve are $(0, \pm 1)$

Implicit differentiation:

$$2x + 3y + (3x + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 3y}{3x + 2y}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,1)} = -\frac{3}{2} \text{ so the tangent has equation } y = 1 - \frac{3}{2}x$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,-1)} = -\frac{3}{2} \text{ so the tangent has equation } y = -1 - \frac{3}{2}x$$

17 **a**

$y^3 - y - x = 0$
When $x = 0$, $y^3 - y = 0$ so $y(y^2 - 1) = 0$: $y = 0, \pm 1$
y-intercepts are $(0,0)$, $(0,1)$ and $(0,-1)$

b

Implicit differentiation:

$$(3y^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3y^2 - 1}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,1)} = \frac{1}{2}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,0)} = -1$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,-1)} = \frac{1}{2}$$

18 **a**

$\mathrm{e}^y - y = x^2$
When $y = 0$, $x^2 = 1$ so $x = \pm 1$
x-intercepts are $(1, 0)$ and $(-1, 0)$

b

Implicit differentiation:

$$(\mathrm{e}^y - 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{\mathrm{e}^y - 1}$$

When $y = 0$, the tangent will be vertical so the two tangent lines are $x = 1$ and $x = -1$

19 **a**

$x^2 - 5xy + y^2 = 1$.
When $x = 1$, $y^2 - 5y = 0$ so $y = 0$ or 5

b

Implicit differentiation:

$$2x - 5y + (2y - 5)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{5y - 2x}{2y - 5}$$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{-2}{-5} = \frac{2}{5}$$ so the tangent equation is $y = \frac{2}{5}(x - 1)$

$$\left.\frac{dy}{dx}\right|_{(1,5)} = \frac{23}{5}$$ so the tangent equation is $(y - 5) = \frac{23}{5}(x - 1)$, which simplifies to $y = \frac{23}{5}x + \frac{2}{5}$

20

Implicit differentiation:

$$e^y\frac{dy}{dx} - \sin y - x\cos y\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}(e^y - x\cos y - y^{-1}) = \sin y$$

$$\frac{dy}{dx} = \frac{\sin y}{e^y - x\cos y - y^{-1}} = \frac{y\sin y}{ye^y - xy\cos y - 1}$$

21

Implicit differentiation:

$$x\cos x + \sin x = (y\cos y + \sin y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x\cos x + \sin x}{y\cos y + \sin y}$$

22

Implicit differentiation:

$$2x + 2y\frac{dy}{dx} = 0 \ (1)$$

so $\frac{dy}{dx} = -\frac{x}{y} \ (2)$

Implicit differentiation again, applied to (1)

$$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

Substituting (2):

$$2 + \frac{2x^2}{y^2} + 2y\frac{d^2y}{dx^2} = 0$$

$$\frac{2(x^2 + y^2)}{y^2} + 2y\frac{d^2y}{dx^2} = 0$$

Substituting $x^2 + y^2 = 9$ and dividing through by 2:

$$\frac{9}{y^2} + y\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{9}{y^3}$$

23

Implicit differentiation:

$2x + 4y + (4x + 4y)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x + 2y}{2x + 2y}$

At stationary point, $\frac{dy}{dx} = 0$ so $x = -2y$

Substituting this into the original equation:

$(-2y)^2 + 4(-2y)y + 2y^2 + 1 = 0$

$-2y^2 + 1 = 0$

$y = \pm\frac{\sqrt{2}}{2}$

Stationary points are $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right)$

24

Implicit differentiation:

$(-3y^2 + 6xy)\frac{dy}{dx} + 3y^2 - 3x^2 = 0$

$\frac{dy}{dx} = \frac{x^2 - y^2}{-y^2 + 2xy}$

Turning points occur when $\frac{dy}{dx} = 0$ so $x^2 = y^2$

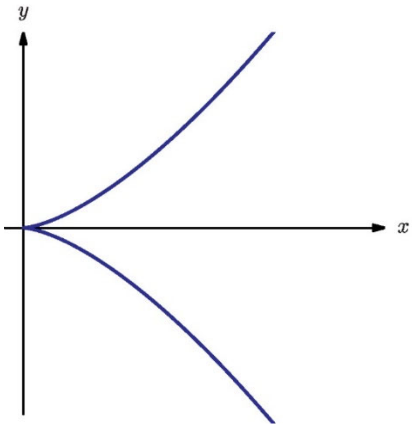
Substituting $x = -y$ into the original equation:

$3y^3 = 8$ so $y = \frac{-2}{\sqrt[3]{3}} = \frac{-2}{3}\sqrt[3]{9}$, and the turning point is $\left(\frac{-2}{3}\sqrt[3]{9}, \frac{-2}{3}\sqrt[3]{9}\right)$

Substituting $x = y$ into the original equation:

$-y^3 = 8$ so $y = -2$, and the turning point is $(-2, -2)$

25 **a**



b

Implicit differentiation:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2y}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(4,8)} = \frac{3(16)}{2(8)} = 3$$

The tangent at $(4, 8)$ has equation $y - 8 = 3(x - 4)$ which simplifies to $y = 3x - 4$

c

Substituting this into the original equation:

$$(3x - 4)^2 = x^3$$

$$x^3 - 9x^2 + 24x - 16 = 0$$

Since there is a tangent intersection at $(4, 8)$, it follows that $(x - 4)^2$ must be a factor of this cubic.

$$x^3 - 9x^2 + 24x - 16 = (x - a)(x - 4)^2$$

Comparison of the constant term shows that $a = 1$

The tangent meets the curve again when $x = 1$; from the tangent equation, this is at $(1, -1)$

Exercise 10D

5

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= 6 - 8\\ &= -2\end{aligned}$$

6

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= 3 - 24\\ &= -21\end{aligned}$$

7

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} &= y^{-1}\frac{\mathrm{d}x}{\mathrm{d}t} - xy^{-2}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= \frac{3}{4} + \frac{3}{16}\\ &= \frac{15}{16}\end{aligned}$$

8

If the side length is x then the area of the square is $A = x^2$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

When $x = 5$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 20\ \mathrm{cm}^2\ \mathrm{s}^{-1}$

9

If the radius is r then the area is $A = \pi r^2$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$$

When $r = 3$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 1.2, \frac{\mathrm{d}A}{\mathrm{d}t} = 7.2\pi \approx 22.6\ \mathrm{mm}^2$ per day

10

If the radius is r then the volume is $V = \frac{4}{3}\pi r^3$ so $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi}\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}}\frac{\mathrm{d}V}{\mathrm{d}t}$$

When $V = 100$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = 200, \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{50}{\pi}\left(\frac{75}{\pi}\right)^{-\frac{2}{3}} \approx 1.92\ \mathrm{cm}^2\ \mathrm{s}^{-1}$

11 Area $A = xy$ and diagonal $d = \sqrt{x^2 + y^2}$

a

$$\frac{\mathrm{d}A}{\mathrm{d}t} = y\frac{\mathrm{d}x}{\mathrm{d}t} + x\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= 10\ \mathrm{cm\ s}^{-1}$$

b

$$\frac{\mathrm{d}d}{\mathrm{d}t} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

$$= \frac{1}{10}(24 - 16)$$

$$= 0.8\ \mathrm{cm\ s}^{-1}$$

12 **a**

The ratio of height to radius is 6 : 1 for the cone, so for any partial filling, $h = 6r$

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{108}$$

b

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 5 = \frac{\pi h^2}{36}\frac{\mathrm{d}h}{\mathrm{d}t}$$

When $h = 18, \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{180}{\pi(18)^2} = \frac{5}{9\pi} \approx 0.177\ \mathrm{cm\ s}^{-1}$

13

$A = \pi r^2$ so $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$

When $\frac{\mathrm{d}A}{\mathrm{d}t} = 86.5, \frac{\mathrm{d}r}{\mathrm{d}t} = 1.8$ so $r = \frac{86.5}{2\pi(1.8)} \approx 7.65$ cm

14

Let the height of the ball above the sportsman be y and the distance horizontally from the sportsman be x.

Then the distance from the sportsman is $d = \sqrt{x^2 + y^2}$

$$\dot{d} = \frac{1}{2\sqrt{x^2 + y^2}}(2x\dot{x} + 2y\dot{y})$$

When $x = 4, y = 2, \dot{x} = 3, \dot{y} = 0$

$$\dot{d} = \frac{1}{2\sqrt{20}}(24) \approx 2.68 \text{ m s}^{-1}$$

15

If density is d, mass m and volume V: $d = mV^{-1}$

$\dot{d} = V^{-1}\dot{m} - mV^{-2}\dot{V}$

When $mV^{-1} = 5, \dot{m} = -2, \dot{V} = -1$

Then $\dot{d} = -2V^{-1} + mV^{-2} = V^{-1}(mV^{-1} - 2) = 3V^{-1} > 0$

The density is increasing.

16

Let x be the distance of the foot of the ladder from the wall and y be the height of the top of the ladder above the base of the wall.

$x^2 + y^2 = 9$ by Pythagoras theorem

Implicit differentiation:

$2x\dot{x} + 2y\dot{y} = 0$

$\dot{x} = -x^{-1}y\dot{y}$

When $y = 2, \dot{y} = -0.1$ and $x = \sqrt{5}$: $\dot{x} = \frac{0.2}{\sqrt{5}} \approx 0.0894 \text{ m s}^{-1}$

Exercise 10E

4

$\mathrm{f}'(x) = 2x - 1$

Stationary point at $x = 0.5$: $(0.5, -0.25)$

End values $\mathrm{f}(0) = 0, \mathrm{f}(2) = 2$

Maximum value is $\mathrm{f}(2) = 2$

Minimum value is $\mathrm{f}(0.5) = -0.25$

5

From GDC, local maximum is at $(1, \mathrm{e}^{-1})$

End values $\mathrm{f}(0.5) = 0.5\mathrm{e}^{-0.5} > \mathrm{f}(2) = 2\mathrm{e}^{-2}$

Maximum value is $\mathrm{f}(1) = \mathrm{e}^{-1}$

Minimum value is $\mathrm{f}(2) = 2\mathrm{e}^{-2}$

6

$x = 6 - 2y$

Let $z = xy = 6y - 2y^2$. This is a negative quadratic so has a single local maximum.

$$\frac{\mathrm{d}z}{\mathrm{d}y} = 6 - 4y$$

Stationary point at $y = 1.5$, for which $z = 4.5$

7

$y = 1 - x$

Let $z = x^3 + y^3 = x^3 + (1 - x)^3 = 1 - 3x + 3x^2$

z is a positive quadratic so has a single local minimum.

$z' = -3 + 6x = 0$ at $x = 0.5$

Minimum value of z is $z(0.5) = 0.25$

8

$y = 5x^{-1}$

Let $z = x + y = x + 5x^{-1}$

$z' = 1 - 5x^{-2}$

Stationary value when $x^2 = 5$. $x > 0$ so $x = \sqrt{5}$

$z(\sqrt{5}) = 2\sqrt{5}$ is a local minimum, since z has no upper bound

9 **a**

Height of the box is x

Side length of the base square is $10 - 2x$

Volume $V = x(10 - 2x)^2 = 4x^3 - 40x^2 + 100x$

b

$V' = 12x^2 - 80x + 100 = 4(3x^2 - 20x + 25) = 4(3x - 5)(x - 5)$

The volume is clearly minimal (zero) if $x = 5$ so the maximal volume is for $x = \frac{5}{3}$

c

Maximum volume $= \frac{5}{3}(10 - \frac{10}{3})^2 = \frac{2000}{27}$

d

Minimal volume is 0 for $x = 0$ or $x = 5$

$x = 0$ is not a stationary point of the cubic but this is because the equation allows for negative values of x, which the context would not permit.

10

Let x be the side length of the square base and h be the height.

Then $V = x^2h = 64$ so $h = 64x^{-2}$

Surface area $A = 2x^2 + 4xh = 2x^2 + 256x^{-1}$

$A' = 4x - 256x^{-2}$

$A' = 0$ if $x^3 = 64$ so $x = 4$

Stationary value is a minimum of the function: When $x = h = 4$, $A = 96\text{ cm}^2$

(The area has no upper bound, since x has no upper bound)

11 **a**

If the height of the cone is h and the radius is r then $h = 10\cos\theta$ and $r = 10\sin\theta$

$$V = \frac{1}{3}\pi r^2 h = \frac{100\pi}{3}\sin^2\theta\cos\theta$$

$$\dot{V} = \frac{100\pi}{3}(2\sin\theta\cos^2\theta - \sin^3\theta)$$

When $\theta = \frac{\pi}{6}, \dot{\theta} = 0.01$ so $\dot{V} = \frac{\pi}{3}\left(\frac{3}{4} - \frac{1}{8}\right) = \frac{5\pi}{24} \approx 0.654\text{ cm}^3\text{ s}^{-1}$

b

Since $V(0) = V\left(\frac{\pi}{2}\right) = 0$, a stationary point for $0 \leq \theta \leq \frac{\pi}{2}$ must be maximal.
$\dot{V} = 0$ when $2\sin\theta\cos^2\theta - \sin^3\theta = 0$
$\sin\theta\,(2\cos^2\theta - \sin^2\theta) = 0$
$\sin\theta = 0$ or $\tan^2\theta = 2$
$\theta = 0$ (minimal volume) or $\theta = \arctan(\sqrt{2}) \approx 0.955$
Since $\theta = \frac{\pi}{6} + 0.01t$, this occurs at $t = 100\left(0.955 - \frac{\pi}{6}\right) = 43.2$ s

12

Let the two base vertices be $(x, 0)$ and $(\pi - x, 0)$
The height of the rectangle is $\sin x$
The area $A = (\pi - 2x)\sin x$
$A' = (\pi - 2x)\cos x - 2\sin x$
From GDC, the maximum in $0 \leq x \leq \pi$ is at $(0.71, 1.12)$
The maximum area is 1.12

13

Let d be the distance from $(1, 2)$ to the point (x, x^3) on the curve.
$d = \sqrt{(x-1)^2 + (x^3 - 2)^2}$
From GDC, the minimum occurs at $x \approx 1.25$, $d \approx 0.254$

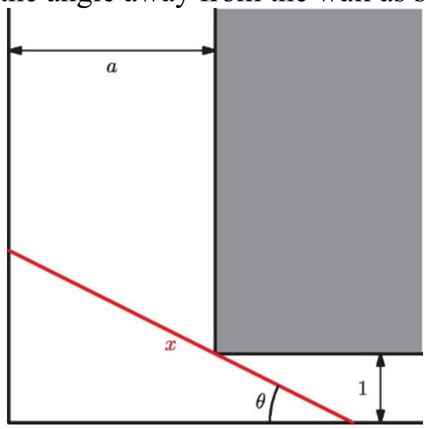
14

Let $2L$ be the total length of the wire.
Let $2x$ be the length of the base of the isosceles, so that each leg of the isosceles has length $L - x$
The area is then $A = x\sqrt{(L-x)^2 - x^2} = x\sqrt{L^2 - 2Lx}$
$A' = \sqrt{L^2 - 2Lx} - \frac{xL}{\sqrt{L^2 - 2Lx}} = \frac{L^2 - 2Lx - xL}{\sqrt{L^2 - 2Lx}}$
$A' = 0$ when $L^2 = 3Lx \Rightarrow x = \frac{L}{3}$
That is, the base of the isosceles triangle must be one third of the total perimeter; the triangle must be equilateral.
Clearly the area is minimal for $x = 0$ or $x = L$, either of which reduce the area to zero.
The stationary point must therefore represent a local maximum for area.

15

The width of one corridor is 1 and the width of the other corridor is a.

Let $x(\theta)$ be the length of a line crossing the corner touching the inner wall, where θ is the angle away from the wall as shown in the diagram:



Restrict θ to only consider $0 < \theta < \frac{\pi}{2}$

x can be split into two parts:

The part in the lower corridor is $x_L = \csc\theta$

The part in the upper corridor is $x_u = a\,\text{cosec}(90° - \theta) = a\sec\theta$

$x = \text{cosec}\,\theta + a\sec\theta$

The longest ladder that can fit around the corner is the minimum value of x as θ varies.

a

$a = 1$

From GDC, minimum x is $2\sqrt{2} \approx 2.83$ m

(This is also obvious from symmetry; minimum x occurs when $\theta = \frac{\pi}{4}$)

b

$a = 8$

From GDC, minimum $x \approx 11.2$ m

Analytically, using the derivatives of trigonometric reciprocal functions shown in the table at the top of the next section:

$$\frac{dx}{d\theta} = -\text{cosec}\,\theta\cot\theta + 8\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{1}{8}$$

$$\tan\theta = \frac{1}{2}$$

$$\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}} \text{ so } x = \sqrt{5} + 4\sqrt{5} = 5\sqrt{5} \approx 11.2$$

Exercise 10F

24

$y' = 4\sec^2 x \tan x$

$y'\left(\frac{\pi}{4}\right) = 4\left(\sqrt{2}\right)^2(1) = 8$

$y\left(\frac{\pi}{4}\right) = 2\left(\sqrt{2}\right)^2 = 4$

Tangent has equation $y - 4 = 8\left(x - \frac{\pi}{4}\right)$

$y = 8x - 2\pi + 4$

25

$y' = 2\sec^2 2x$

$y'\left(\frac{\pi}{6}\right) = 2(2)^2 = 8$ so normal has gradient $-\frac{1}{8}$

$y\left(\frac{\pi}{6}\right) = \sqrt{3}$

Normal has equation $y - \sqrt{3} = -\frac{1}{8}\left(x - \frac{\pi}{3}\right)$

$y = \sqrt{3} = -\frac{1}{8}x + \frac{\pi}{48}$

26

$y' = \frac{2}{\sqrt{1-x^2}}$

$y'(0) = 2$ so the gradient of the graph where it crosses the y-axis is 2.

27

$y' = \frac{3}{2\left(1+\left(\frac{x}{2}\right)^2\right)} = \frac{6}{4+x^2}$

28 **a**

$y = \tan x + (\tan x)^{-1}$

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$ so using chain rule:

$$\begin{aligned} y' &= \sec^2 x - \frac{\sec^2 x}{\tan^2 x} \\ &= \sec^2 x - \operatorname{cosec}^2 x \\ &= \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} \\ &= -\frac{\cos 2x}{\frac{1}{4}\sin^2 2x} \\ &= -4\cot 2x \operatorname{cosec} 2x \end{aligned}$$

b

$y' = 0$: $\cot 2x = 0$ or $\operatorname{cosec} 2x = 0$ (no real solutions)

$\cot 2x = 0$: $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Stationary points are $\left(\frac{\pi}{4}, 2\right)$ and $\left(\frac{3\pi}{4}, -2\right)$

29

$y = 3^x$

$y' = 3^x \ln 3$

$3^x \ln 3 = \ln 81$

$3^x = \frac{\ln(81)}{\ln(3)} = \ln_3 81 = 4$

$x = \log_3 4$

Coordinates are $(\log_3 4, 4)$

30

$$\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx = \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{2}$$

31

$$\int_0^3 2^x \, dx = \left[\frac{1}{\ln 2} 2^x\right]_0^3 = \frac{7}{\ln 2}$$

32

$$\int_0^a \frac{6}{\sqrt{1-x^2}} \, dx = \pi = [6 \arcsin x]_0^a = 6 \arcsin a$$

$a = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

33

$y' = 3\pi \sec^2(\pi x)$

$y = \int 3\pi \sec^2(\pi x) \, dx = 3\tan(\pi x) + c$

$y\left(\frac{1}{4}\right) = 5 = 3 + c \Rightarrow c = 2$

$y = 3\tan(\pi x) + 2$

34

$f(x) = \tan x - \cot x$

$f'(x) = \sec^2 x + \mathrm{cosec}^2 x$

The derivative is the sum of two squared functions with finite values in the interval $0 < x < \frac{\pi}{2}$, so is always positive.

By definition, since $f'(x) > 0$ in the interval, $f(x)$ is increasing.

35 **a**

$\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$ for some constants A and B

Multiplying by the denominator on the LHS:

$3 = A(x+1) + B(x-2)$

Substituting:

$x = -1: 3 = -3B \Rightarrow B = -1$

$x = 2: 3 = 3A \Rightarrow A = 1$

$$\frac{3}{(x-2)(x+1)} = \frac{1}{x-2} - \frac{1}{x+1}$$

b

$$\int \frac{3}{(x-2)(x+1)}\,\mathrm{d}x = \int \frac{1}{x-2} - \frac{1}{x+1}\,\mathrm{d}x$$
$$= \ln|x-2| - \ln|x+1| + c$$
$$= \ln\left|\frac{x-2}{x+1}\right| + c$$

36 **a**

$$\frac{x-6}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$ for some constants A and B

Multiplying by the denominator on the LHS:

$x - 6 = A(x+2) + B(x-2)$

Substituting:

$x = -2: -8 = -4B \Rightarrow B = 2$

$x = 2: -4 = 4A \Rightarrow A = -1$

$$\frac{x-6}{(x-2)(x+2)} = \frac{2}{x+2} - \frac{1}{x-2}$$

b

$$\int_0^1 \frac{x-6}{(x-2)(x+2)}\,\mathrm{d}x = \int_0^1 \frac{2}{x+2} - \frac{1}{x-2}\,\mathrm{d}x$$
$$= [2\ln|x+2| - \ln|x-2|]_0^1$$
$$= (2\ln 3 - 0) - (2\ln 2 - \ln 2)$$
$$= 2\ln 3 - \ln 2$$
$$= \ln\left(\frac{9}{2}\right)$$

37

$$\tan x = \frac{\sin x}{\cos x}$$

Let $u = \sin x, v = \cos x$ so $u' = \cos x, v' = -\sin x$

Quotient Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Then $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

38

$$\operatorname{cosec} x = \frac{1}{\sin x} = (\sin x)^{-1}$$

By Chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec} x = -\cos x\,(\sin x)^{-2} = -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = -\cot x \operatorname{cosec} x$$

39

$\cot x = \dfrac{\cos x}{\sin x}$

Let $u = \cos x, v = \sin x$ so $u' = -\sin x, v' = \cos x$

Quotient Rule:

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right) = \dfrac{u'v - uv'}{v^2}$

Then $\dfrac{\mathrm{d}}{\mathrm{d}x}\cot(x) = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \dfrac{-1}{\sin^2 x} = -\mathrm{cosec}^2\, x$

40

$\mathrm{f}(x) = \log_a x = \dfrac{\ln x}{\ln a}$ using the change of base rule

Then $\mathrm{f}'(x) = \dfrac{1}{\ln a}\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \dfrac{1}{x \ln a}$

41

Let $y = \arccos x$ so $x = \cos y$

Implicit differentiation:

$1 = -\sin y\, y'$

So $y' = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$

42

Let $y = \arctan x$ so $x = \tan y$

Implicit differentiation:

$1 = \sec^2 y\ y'$

So $y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$

43

Let V be the volume of water in the container.

$\dot{V} = \dfrac{1}{1+4t^2}$

$$\begin{aligned} V(0.25) &= \int_0^{0.25} \frac{1}{1+4t^2}\,\mathrm{d}t + V(0) \\ &= \left[\frac{1}{2}\arctan(2t)\right]_0^{0.25} + 120 \\ &= 231.8 + 120 \\ &\approx 352 \text{ ml} \end{aligned}$$

44 **a**

Using Chain Rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x^{0.5}) &= 0.5x^{-0.5} \times \frac{1}{\sqrt{1-x}} \\ &= \frac{1}{2\sqrt{x-x^2}} \end{aligned}$$

b

$$\int_0^1 \frac{1}{\sqrt{x-x^2}}\,\mathrm{d}x = \left[2\arcsin\sqrt{x}\right]_0^1 = \pi$$

45 **a**

Using Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(\sec x)) = \frac{\sec x \tan x}{\sec x} = \tan x$$

b

$$\int_0^{\frac{\pi}{9}} 2\tan 3x\ \mathrm{d}x = \left[\frac{2}{3}\ln(\sec 3x)\right]_0^{\frac{\pi}{9}}$$
$$= \frac{2}{3}(\ln 2 - \ln 1)$$
$$= \frac{2}{3}\ln 2$$

46

If you have the result from question **44a** above, you can directly integrate this function; the solution below does not use this knowledge, and so completing the square is the sensible approach to convert the denominator into a form which can be used with a standard formula for integration.

$6x - x^2 = 9 - (x-3)^2$

$$\int_3^{4.5} \frac{1}{\sqrt{6x-x^2}}\,\mathrm{d}x = \int_3^{4.5} \frac{1}{\sqrt{9-(x-3)^2}}\,\mathrm{d}x$$
$$= \left[\arcsin\left(\frac{x-3}{3}\right)\right]_3^{4.5}$$
$$= \arcsin\left(\frac{1}{2}\right) - \arcsin(0)$$
$$= \frac{\pi}{6}$$

47 **a**

$\dfrac{1}{(x+1)(x+5)} = \dfrac{A}{x+1} + \dfrac{B}{x+5}$ for some constants A and B

Multiplying by the denominator on the LHS:

$1 = A(x+5) + B(x+1)$

Substituting:

$x = -1: 1 = 4A \Rightarrow A = \dfrac{1}{4}$

$x = -5: 1 = -4B \Rightarrow B = -\dfrac{1}{4}$

$$\frac{1}{(x+1)(x+5)} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x+5}\right)$$

$$\int \frac{1}{(x+1)(x+5)}\,\mathrm{d}x = \frac{1}{4}(\ln|x+1| - \ln|x+5|) + c$$
$$= \frac{1}{4}\ln\left|\frac{x+1}{x+5}\right| + c$$

b

$$\frac{1}{x^2+6x+18} = \frac{1}{(x+3)^2+9}$$

$$\int \frac{1}{(x+3)^2+9}\,\mathrm{d}x = \frac{1}{3}\arctan\left(\frac{x+3}{3}\right) + c$$

c)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+6x+18) = 2x+6$$

$$\text{Then, since } \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|f(x)| + c\text{:}$$

$$\int \frac{2x}{x^2+6x+18}\mathrm{d}x = \int \frac{2x+6}{x^2+6x+18} - \frac{6}{x^2+6x+18}\mathrm{d}x$$

$$= \ln|x^2+6x+18| - 2\arctan\left(\frac{x+3}{3}\right) + c$$

48

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

Then by Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x+b}{a}\right)\right) = \frac{1}{a} \times \frac{1}{\sqrt{1-\left(\frac{x+b}{a}\right)^2}} = \frac{1}{\sqrt{a^2-(x-b)^2}}$$

It then follows, since the indefinite integral equals the antiderivative plus a constant,

$$\int \frac{1}{\sqrt{a^2-(x+b)^2}}\mathrm{d}x = \arcsin\left(\frac{x+b}{a}\right) + c$$

49 **a**

$$4x^2-8x+29 = 4(x^2-2x)+29$$
$$= 4((x-1)^2-1)+29$$
$$= 4(x-1)^2+25$$
$$= (2x-2)^2+25$$

b

$$\int \frac{1}{4x^2-8x+19}\,\mathrm{d}x = \int \frac{1}{5^2+(2x-2)^2}\,\mathrm{d}x$$
$$= \frac{1}{5\times 2}\arctan\left(\frac{2x-2}{5}\right) + c$$
$$= \frac{1}{10}\arctan\left(\frac{2x-2}{5}\right) + c$$

50 **a**

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\arcsin x) = \arcsin x + \frac{x}{\sqrt{1-x^2}}$$

b

From part **a**,

$$\arcsin x = \frac{\mathrm{d}}{\mathrm{d}x}(x\arcsin x) - \frac{x}{\sqrt{1-x^2}}$$

Then

$$\int \arcsin x\;\mathrm{d}x = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$= x\arcsin x + \sqrt{1-x^2} + c$$

51

Using L'Hôpital's Rule:

$$\lim_{x\to 0}\left\{\frac{\arctan x}{x}\right\} = \lim_{x\to 0}\left\{\frac{\left(\frac{1}{1+x^2}\right)}{1}\right\} = 1$$

52

Using L'Hôpital's Rule:

$$\lim_{x\to 0}\left\{\frac{2^x - 1}{x}\right\} = \lim_{x\to 0}\left\{\frac{2^x \ln 2}{1}\right\} = \ln 2$$

Exercise 10G

10

Let $u = x - 2$ so $\mathrm{d}u = \mathrm{d}x$

Limits:

$x = 2: u = 0$

$x = 3: u = 1$

$$\begin{aligned}\int_2^3 x(x-2)^3\,\mathrm{d}x &= \int_0^1 (u+2)u^3\ \mathrm{d}u\\ &= \int_0^1 u^4 + 2u^3\ \mathrm{d}u\\ &= \left[\frac{1}{5}u^5 + \frac{1}{2}u^4\right]_0^1\\ &= \frac{7}{10}\end{aligned}$$

11

Let $u = x + 1$ so $\mathrm{d}u = \mathrm{d}x$

Limits:

$x = 0: u = 1$

$x = 1: u = 2$

$$\begin{aligned}\int_0^1 x^2\sqrt{x+1}\,\mathrm{d}x &= \int_1^2 (u-1)^2\sqrt{u}\ \mathrm{d}u\\ &= \int_0^1 u^{2.5} - 2u^{1.5} + u^{0.5}\ \mathrm{d}u\\ &= \left[\frac{2}{7}u^{3.5} - \frac{4}{5}u^{2.5} + \frac{2}{3}u^{1.5}\right]_1^2\\ &= \left(\frac{2}{7}\times 8\sqrt{2} - \frac{4}{5}\times 4\sqrt{2} + \frac{2}{3}\times 2\sqrt{2}\right) - \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right)\\ &= \frac{240 - 336 + 140}{105}\sqrt{2} - \frac{30 - 84 + 70}{105}\\ &= \frac{44\sqrt{2} + 16}{105}\end{aligned}$$

12

Let $u = x + 2$ so $\mathrm{d}u = \mathrm{d}x$

Limits:

$x = 0: u = 2$

$x = 7: u = 9$

$$\int_0^7 \frac{x}{\sqrt{x+2}}\,\mathrm{d}x = \int_2^9 \frac{u-2}{\sqrt{u}}\,\mathrm{d}u$$
$$= \int_2^9 u^{0.5} - 2u^{-0.5}\,\mathrm{d}u$$
$$= \left[\frac{2}{3}u^{1.5} - 4u^{0.5}\right]_2^9$$
$$= \left(\frac{2}{3}\times 27 - 4\times 3\right) - \left(\frac{2}{3}\times 2\sqrt{2} - 4\times\sqrt{2}\right)$$
$$= 6 + \frac{8\sqrt{2}}{3}$$

13

Let $x = u^2$ so $\mathrm{d}x = 2u\,\mathrm{d}u$

$$\int \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int \frac{\mathrm{e}^u}{u}\times 2u\,\mathrm{d}u$$
$$= \int 2\mathrm{e}^u\,\mathrm{d}u$$
$$= 2\mathrm{e}^u + c$$
$$= 2\mathrm{e}^{\sqrt{x}} + c$$

14

Let $u = x + 4$ so $\mathrm{d}u = \mathrm{d}x$

Limits:

$x = 0: u = 4$

$x = 5: u = 9$

$$\int_0^5 \frac{x}{\sqrt{x+4}}\,\mathrm{d}x = \int_4^9 \frac{u-4}{\sqrt{u}}\,\mathrm{d}u$$
$$= \int_4^9 u^{0.5} - 4u^{-0.5}\,\mathrm{d}u$$
$$= \left[\frac{2}{3}u^{1.5} - 8u^{0.5}\right]_4^9$$
$$= \left(\frac{2}{3}\times 27 - 8\times 3\right) - \left(\frac{2}{3}\times 8 - 8\times 2\right)$$
$$= \frac{14}{3}$$

15 **a**

$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ for some constants A and B

Multiplying by the denominator on the LHS:

$1 = A(u+1) + Bu$

Substituting:

$x = 0: 1 = A \Rightarrow A = 1$

$x = -1: 1 = -B \Rightarrow B = -1$

$\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}$

b

Let $u = \mathrm{e}^x$ so $\mathrm{d}u = \mathrm{e}^x\ \mathrm{d}x$

$$\begin{aligned}
\int \frac{1}{\mathrm{e}^x+1}\ \mathrm{d}x &= \int \frac{1}{\mathrm{e}^x(\mathrm{e}^x+1)}\ \mathrm{e}^x\ \mathrm{d}x \\
&= \int \frac{1}{u(u+1)}\ \mathrm{d}u \\
&= \int \frac{1}{u} - \frac{1}{u+1}\ \mathrm{d}u \\
&= \ln|u| - \ln|u+1| + c \\
&= \ln\left|\frac{u}{u+1}\right| + c \\
&= \ln\left|\frac{e^x}{e^x+1}\right| + c \\
&= -\ln|1+\mathrm{e}^{-x}| + c
\end{aligned}$$

16

Let $u = \mathrm{e}^x - 1$ so $\mathrm{d}u = \mathrm{e}^x\ \mathrm{d}x$

Limits:

$x = \ln 3 : u = 2$

$x = \ln 5 : u = 4$

$$\begin{aligned}
\int_{\ln 3}^{\ln 5} \frac{\mathrm{e}^{2x}}{\mathrm{e}^x-1} \mathrm{d}x &= \int_{\ln 3}^{\ln 5} \frac{\mathrm{e}^x}{\mathrm{e}^x-1} \mathrm{e}^x\ \mathrm{d}x \\
&= \int_2^4 \frac{u+1}{u}\ \mathrm{d}u \\
&= \int_2^4 1 + u^{-1}\ \mathrm{d}u \\
&= [u + \ln|u|]_2^4 \\
&= (4 + \ln 4) - (2 + \ln 2) \\
&= 2 + \ln 2
\end{aligned}$$

17

Let $u = \ln x$ so $\mathrm{d}u = x^{-1}\ \mathrm{d}x$

$$\begin{aligned}
\int (\ln x)^2 x^{-1}\ \mathrm{d}x &= \int u^2\ \mathrm{d}u \\
&= \frac{1}{3}u^3 + c \\
&= \frac{1}{3}(\ln x)^3 + c
\end{aligned}$$

18

Let $u = \tan x$ so $\mathrm{d}u = \sec^2 x \ \mathrm{d}x$

$$\begin{aligned}\int \sec^4 x \ \mathrm{d}x &= \int \sec^2 x \sec^2 x \ \mathrm{d}x \\ &= \int (1 + \tan^2 x)\sec^2 x \ \mathrm{d}x \\ &= \int (1 + u^2) \ \mathrm{d}u \\ &= u + \frac{1}{3}u^3 + c \\ &= \tan x + \frac{1}{3}\tan^3 x + c\end{aligned}$$

19

Let $x = \ln(\sec u)$ so $u = \arccos(\mathrm{e}^{-x})$ and $\mathrm{d}u = \dfrac{\mathrm{e}^{-x}}{\sqrt{1 - \mathrm{e}^{-2x}}} \mathrm{d}x$

$$\begin{aligned}\int \frac{1}{\sqrt{\mathrm{e}^{2x} - 1}} \mathrm{d}x &= \int \frac{\mathrm{e}^{-x}}{\sqrt{1 - \mathrm{e}^{-2x}}} \mathrm{d}x \\ &= \int 1 \, \mathrm{d}u \\ &= u + c \\ &= \arccos(\mathrm{e}^{-x}) + c\end{aligned}$$

20

Let $u = \sin x$ so $\mathrm{d}u = \cos x \ \mathrm{d}x$

Limits:

When $x = 0, u = 0$

When $x = \dfrac{\pi}{6}, u = \dfrac{1}{2}$

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \frac{3\cos x}{10 - \cos^2 x} \mathrm{d}x &= \int_0^{\frac{\pi}{6}} \frac{3}{9 + \sin^2 x} \cos x \ \mathrm{d}x \\ &= \int_0^{\frac{1}{2}} \frac{3}{9 + u^2} \mathrm{d}u \\ &= \left[\frac{3}{3}\arctan\left(\frac{u}{3}\right)\right]_0^{\frac{1}{2}} \\ &= \arctan\left(\frac{1}{6}\right)\end{aligned}$$

21

Let $u = 1 + \mathrm{e}^x$ so $\mathrm{d}u = \mathrm{e}^x \ \mathrm{d}x$

$$\begin{aligned}\int \frac{1}{1 + \mathrm{e}^x} \mathrm{d}x &= \int \frac{1}{\mathrm{e}^x + \mathrm{e}^{2x}} \mathrm{e}^x \ \mathrm{d}x \\ &= \int \frac{1}{(u - 1)u} \mathrm{d}u\end{aligned}$$

Partial Fractions:
$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \text{ for some constants } A \text{ and } B$$
Multiplying through by the denominator on the LHS:
$$1 = A(u-1) + Bu$$
Substituting:
$$u = 0: 1 = -A \Rightarrow A = -1$$
$$u = 1: 1 = B$$
$$\frac{1}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u}$$
$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u-1} - \frac{1}{u}\,du$$
$$= \ln|u-1| - \ln|u| + c$$
$$= x - \ln|1+e^x| + c$$

22

Let $x = \frac{2\sqrt{2}}{5}\sin u$ so $dx = \frac{2\sqrt{2}}{5}\cos u\,du$ and $u = \arcsin\left(\frac{5x}{2\sqrt{2}}\right)$
$$\int \frac{1}{\sqrt{8-25x^2}}\,dx = \frac{1}{2\sqrt{2}}\int \frac{1}{\sqrt{1-\frac{25}{8}x^2}}\,dx$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{1-\sin^2 u}}\cos u\,du$$
$$= \frac{1}{5}\int \frac{1}{\cos u}\cos u\,du$$
$$= \frac{1}{5}u + c$$
$$= \frac{1}{5}\arcsin\left(\frac{5x}{2\sqrt{2}}\right) + c$$

23

Let $x = \sin u$ so $dx = \cos u\,du$

Limits:
$$x = 0: u = 0$$
$$x = 1: u = \frac{\pi}{2}$$
$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2 u\,du$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2u + 1)\,du$$
$$= \left[\frac{1}{4}\sin 2u + \frac{1}{2}u\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4}$$

Tip: Alternatively, as hinted in the book answers, realise that this integral calculates the area of the first quadrant of the unit square, whose equation is given by $x^2 + y^2 = 1$, so that $y = \sqrt{1-x^2}$ in the upper two quadrants. The area is clearly one quarter of the area of the circle, π.

24 **a**

$x = e^u - e^{-u}$

$(e^u)^2 - xe^u - 1 = 0$

$e^u = \dfrac{x \pm \sqrt{x^2+4}}{2}$

Since $e^u > 0$ for all real u, only the positive root is valid.

$e^u = \dfrac{x + \sqrt{x^2+4}}{2}$

b

Let $x = e^u - e^{-u}$ so $dx = (e^u + e^{-u})\,du$

Limits:

$x = 0: e^u = 1$ so $u = 0$

$x = 1: e^u = \dfrac{1+\sqrt{5}}{2}$ so $u = \ln\left(\dfrac{1+\sqrt{5}}{2}\right)$.

For simplicity of algebra in the working, let $a = \ln\left(\dfrac{1+\sqrt{5}}{2}\right)$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{4+x^2}}\,dx &= \int_0^a \frac{1}{\sqrt{e^{2u}+2+e^{-2u}}}(e^u+e^{-u})\,du\\ &= \int_0^a \frac{e^u+e^{-u}}{\sqrt{(e^u+e^{-u})^2}}\,du\\ &= \int_0^a 1\,du\\ &= [u]_0^a\\ &= a\\ &= \ln\left(\frac{1+\sqrt{5}}{2}\right)\end{aligned}$$

Exercise 10H

10

Let $u = x, v' = e^{2x}$ so $u' = 1, v = \dfrac{1}{2}e^{2x}$

Integration by parts: $\displaystyle\int uv'\,dx = uv - \int u'v\,dx$

$$\begin{aligned}\int_0^1 xe^{2x}\,dx &= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx\\ &= \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1\\ &= \frac{1}{4}(e^2+1)\end{aligned}$$

11

Let $u = x, v' = \cos x$ so $u' = 1, v = \sin x$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$

$$\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x$$
$$= [x\sin x + \cos x]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2} - 1$$

12

Let $u = 2x, v' = \mathrm{e}^{-3x}$ so $u' = 2, v = -\frac{1}{3}\mathrm{e}^{-3x}$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$

$$\int 2x\mathrm{e}^{-3x}\,\mathrm{d}x = -\frac{2}{3}x\mathrm{e}^{-3x} + \int \frac{2}{3}\mathrm{e}^{-3x}\,\mathrm{d}x$$
$$= -\frac{2}{3}x\mathrm{e}^{-3x} - \frac{2}{9}\mathrm{e}^{-3x} + c$$
$$= -\frac{2}{9}(3x+1)\mathrm{e}^{-3x} + c$$

13

Let $u = \ln x, v' = x^5$ so $u' = x^{-1}, v = \frac{1}{6}x^6$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$

$$\int_1^{\mathrm{e}} x^5\ln x\,\mathrm{d}x = \left[\frac{1}{6}x^6\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{1}{6}x^5\,\mathrm{d}x$$
$$= \left[\frac{1}{6}x^6\ln x - \frac{1}{36}x^6\right]_1^{\mathrm{e}}$$
$$= \frac{1}{6}\mathrm{e}^6 - \frac{1}{36}\mathrm{e}^6 + \frac{1}{36}$$
$$= \frac{1}{36}(5\mathrm{e}^6 + 1)$$

14

Let $u = \ln 2x, v' = x^2$ so $u' = x^{-1}, v = \frac{1}{3}x^3$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$

$$\int_1^2 x^2\ln 2x\,\mathrm{d}x = \left[\frac{1}{3}x^3\ln 2x\right]_1^2 - \int_1^2 \frac{1}{3}x^2\,\mathrm{d}x$$
$$= \left[\frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3\right]_1^2$$
$$= \frac{8}{3}\ln 4 - \frac{8}{9} - \left(\frac{1}{3}\ln 2 - \frac{1}{9}\right)$$
$$= 5\ln 2 - \frac{7}{9}$$

15

Let $u = \ln x$, $v' = x^{-0.5}$ so $u' = x^{-1}$, $v = 2x^{0.5}$

Integration by parts: $\int uv'\,dx = uv - \int u'v\ dx$

$$\int_1^4 x^{-0.5}\ln x\ dx = \left[2x^{\frac{1}{2}}\ln x\right]_1^4 - \int_1^4 2x^{-\frac{1}{2}}\ dx$$
$$= \left[2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}}\right]_1^4$$
$$= 4\ln 4 - 8 - (0 - 4)$$
$$= 8\ln 2 - 4$$

16

Let $u = x^2$, $v'' = e^{-x}$ so $u' = 2x$, $v' = -e^{-x}$, $u'' = 2$, $v = e^{-x}$

Integration by parts (twice): $\int uv''\,dx = uv' - \int u'v'\ dx$

$$= uv' - u'v + \int u''v\ dx$$

$$\int x^2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}\ dx$$
$$= e^{-x}(x^2 + 2x + 2) + c$$

17

Let $u = x^2$, $v'' = \sin x$ so $u' = 2x$, $v' = -\cos x$, $u'' = 2$, $v = -\sin x$

Integration by parts (twice): $\int uv''\,dx = uv' - \int u'v'\ dx$

$$= uv' - u'v + \int u''v\ dx$$

$$\int_0^\pi x^2\sin x\,dx = [-x^2\cos x + 2x\sin x]_0^\pi - \int_0^\pi 2\sin x\ dx$$
$$= [-x^2\cos x + 2x\sin x + 2\cos x]_0^\pi$$
$$= \pi^2 - 2 - 2$$
$$= \pi^2 - 4$$

18

Let $u = \ln x$, $v' = (2x + 1)$ so $u' = x^{-1}$, $v = x^2 + x$

Integration by parts: $\int uv'\,dx = uv - \int u'v\ dx$

$$\int (2x + 1)\ln x\ dx = (x^2 + x)\ln x - \int x + 1\ dx$$
$$= (x^2 + x)\ln x - \frac{1}{2}x^2 - x + c$$

19 **a**

$$\int \sec^2(2x)\ \mathrm{d}x = \frac{1}{2}\tan(2x) + c$$

b

Let $u = x, v' = \sec^2(2x)$ so $u' = 1, v = \frac{1}{2}\tan(2x)$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\ \mathrm{d}x$

$$\begin{aligned}\int x\sec^2(2x)\ \mathrm{d}x &= \frac{1}{2}x\tan(2x) - \int \frac{1}{2}\tan(2x)\ \mathrm{d}x\\ &= \frac{1}{2}x\tan(2x) + \frac{1}{4}\ln\cos(2x) + c\end{aligned}$$

20 **a**

$$\begin{aligned}\int \frac{x^2}{1+x^2}\ \mathrm{d}x &= \int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\ \mathrm{d}x\\ &= \int 1\ \mathrm{d}x - \int \frac{1}{1+x^2}\ \mathrm{d}x\\ &= x - \arctan x + c\end{aligned}$$

b

Let $u = \arctan x\,, v' = x$ so $u' = \frac{1}{1+x^2}, v = \frac{1}{2}x^2$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\ \mathrm{d}x$

$$\begin{aligned}\int x\arctan x\ \mathrm{d}x &= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ \mathrm{d}x\\ &= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x - \arctan x) + c\\ &= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + c\end{aligned}$$

21 **a**

Using Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(\sec x)) = \frac{\sec x\tan x}{\sec x} = \tan x$$

b

Let $u = \ln(\sec x)\,, v' = \sin x$ so $u' = \tan x\,, v = -\cos x$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\ \mathrm{d}x$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x\ln(\sec x)\ \mathrm{d}x &= [-\cos x\ln(\sec x)]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x\tan x\ \mathrm{d}x\\ &= [-\cos x\ln(\sec x) - \cos x]_0^{\frac{\pi}{4}}\\ &= -\frac{\sqrt{2}}{2}\ln(\sqrt{2}) - \frac{\sqrt{2}}{2} + 1\\ &= 1 - \frac{\sqrt{2}}{4}(2 + \ln 2)\end{aligned}$$

22 **a**

Let $u = \cos x, v' = e^x$ so $u' = -\sin x, v = e^x$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$$I = \int e^x \cos x \, dx$$
$$= e^x \cos x + \int e^x \sin x \, dx$$
$$= e^x \cos x + J + c_1$$

Let $u = \sin x, v' = e^x$ so $u' = \cos x, v = e^x$

$$J = \int e^x \sin x \, dx$$
$$= e^x \sin x - \int e^x \cos x \, dx$$
$$= e^x \sin x - I + c_2$$

b

Substituting the second result into the first (and allowing the sum of two unknown constants to be written as one):

$$I = e^x(\cos x + \sin x) - I + c$$
$$I = \frac{1}{2}e^x(\cos x + \sin x) + c$$

23

Let $u = \ln x, v' = 1$ so $u' = x^{-1}, v = x$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$$\int \ln x \, dx = x \ln x - \int 1 \, dx$$
$$= x(\ln x - 1) + c$$

24

Let $u = (\ln x)^2, v' = 1$ so $u' = 2x^{-1} \ln x, v = x$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int 2 \ln x \, dx$$

Using the same working as shown in the previous question,

$$\int \ln x \, dx = x(\ln x - 1) + c$$
$$\int (\ln x)^2 \, dx = x((\ln x)^2 - 2 \ln x + 2) + x$$

25

Let $u = \arctan x$, $v' = 1$ so $u' = \frac{1}{1+x^2}$, $v = x$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$$\begin{aligned}\int \arctan x \, dx &= x \arctan x - \int \frac{x}{1+x^2} \, dx \\ &= x \arctan x - \frac{1}{2}\int \frac{2x}{1+x^2} \, dx \\ &= x \arctan x - \frac{1}{2}\ln(1+x^2) + c\end{aligned}$$

$\left(\text{using that } \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c\right)$

Tip: Note that modulus signs are not needed for the logarithm since $1 + x^2$ is positive for all x

26

Let $u = e^{3x}$, $v'' = \sin 2x$ so $u' = 3e^{3x}$, $v' = -\frac{1}{2}\cos 2x$, $u'' = 9e^{3x}$, $v = -\frac{1}{4}\sin 2x$

Integration by parts (twice): $\int uv'' \, dx = uv' - \int u'v' \, dx$

$$= uv' - u'v + \int u''v \, dx$$

Let $I = \int e^{3x} \sin 2x \, dx$

$$\begin{aligned}I &= e^{3x}\left(-\frac{1}{2}\cos 2x + \frac{3}{4}\sin 2x\right) - \int \frac{9}{4}e^{3x} \sin 2x \, dx \\ &= \frac{e^{3x}}{4}(3\sin 2x - 2\cos 2x) - \frac{9}{4}I + c\end{aligned}$$

$$I = \frac{4}{13}\left(\frac{e^{3x}}{4}(3\sin 2x - 2\cos 2x)\right) = \frac{1}{13}e^{3x}(3\sin 2x - 2\cos 2x) + c$$

27

Let $u = \cos 3x$, $v'' = e^{-x}$ so $u' = -3\sin 3x$, $v' = -e^{-x}$, $u'' = -9\cos 3x$, $v = e^{-x}$

Integration by parts (twice): $\int uv'' \, dx = uv' - \int u'v' \, dx$

$$= uv' - u'v + \int u''v \, dx$$

Let $I = \int e^{-x} \cos 3x \, dx$

$$\begin{aligned}I &= e^{-x}(-\cos 3x + 3\sin 3x) - \int 9e^{-x} \cos 3x \, dx \\ &= e^{-x}(3\sin 3x - \cos 3x) - 9I + c\end{aligned}$$

$$I = \frac{1}{10}e^{-x}(3\sin 3x - \cos 3x) + c$$

28 **a**

Taking $n \neq -1$:
Let $u = x^n, v' = \mathrm{e}^x$ so $u' = nx^{n-1}, v = \mathrm{e}^x$

Integration by parts: $\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$

$$I_n = \int \mathrm{e}^x x^n\,\mathrm{d}x$$
$$= \mathrm{e}^x x^n - \int n\mathrm{e}^x x^{n-1}\,\mathrm{d}x$$
$$= \mathrm{e}^x x^n - nI_{n-1}$$

b

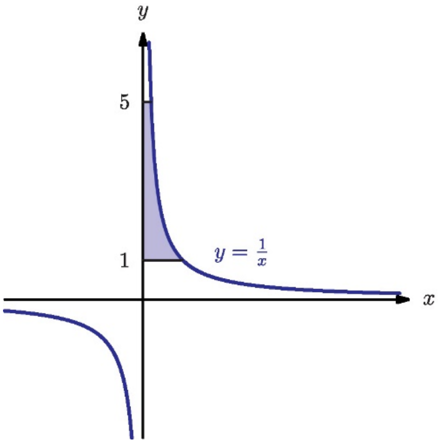
By iteration,
$$I_3 = \mathrm{e}^x x^3 - 3I_2$$
$$= \mathrm{e}^x x^3 - 3(\mathrm{e}^x x^2 - 2I_1)$$
$$= \mathrm{e}^x(x^3 - 3x^2) + 6(\mathrm{e}^x x - I_0)$$
$$I_0 = \int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c$$

So $I_3 = \mathrm{e}^x(x^3 - 3x^2 + 6x - 6) + k$

$$\int_0^1 x^3\mathrm{e}^x\,\mathrm{d}x = [\mathrm{e}^x(x^3 - 3x^2 + 6x - 6)]_0^1$$
$$= \mathrm{e}(1 - 3 + 6 - 6) + 6$$
$$= 6 - 2\mathrm{e}$$

Exercise 10I

13



a

$$y = \frac{1}{x} \text{ so } x = \frac{1}{y}$$
$$\int_{0.2}^{1} y^{-1}\,\mathrm{d}y = [\ln x]_{0.2}^{1} = 0 - \ln\left(\frac{1}{5}\right) = \ln 5$$

b

$$\pi\int_{0.2}^{1} (y^{-1})^2\,\mathrm{d}y = 4\pi$$

14

$$\pi\int_1^a (x^{-1})^2 \, dx = \pi[-x^{-1}]_1^a = \pi\left(1-\frac{1}{a}\right) = \frac{2}{3}\pi$$

$a = 3$

15

$y = x^2$ so $x = \sqrt{y}$

$$\pi\int_0^{a^2} x^2 \, dy = \pi\int_0^{a^2} y \, dy = \pi\left[\frac{1}{2}y^2\right]_0^{a^2} = \frac{1}{2}\pi a^4 = 8\pi$$

$a = \pm 2$

16 **a** x-coordinate of A is -3

b

$$V = \pi\int_{-3}^{3} y^2 \, dx$$

$$= \pi\int_{-3}^{3} x + 3 \, dx$$

$$= 18\pi$$

17 **a**

$$A = \int_{-2}^{2} x \, dy$$

$$= \int_{-2}^{2} 4 - y^2 \, dy$$

$$= \frac{32}{3}$$

b

$$V = \pi\int_{-2}^{2} x^2 \, dy$$

$$= \pi\int_{-2}^{2} 16 - 8y^2 + y^4 \, dy$$

$$= \frac{512\pi}{15}$$

18 **a**

$$A = \int_0^{\pi} y \, dx$$

$$= \int_0^{\pi} \sqrt{x}\sin x \, dx$$

$$= 2.43 \text{ (GDC)}$$

b

$$V = \pi\int_0^{\pi} y^2 \, dx$$

$$= \pi\int_0^{\pi} x\sin^2 x \, dx$$

$$= 7.75 \text{ (GDC)}$$

Tip: Unlike part **a**, this could be calculated exactly by algebra, using integration by parts:

$$\pi \int_0^\pi x \sin^2 x \ \mathrm{d}x = \frac{\pi}{2}\int_0^\pi x(1-\cos 2x)\, dx$$
$$= \frac{\pi}{2}(\int_0^\pi x \ \mathrm{d}x - \int_0^\pi x\cos 2x \ \mathrm{d}x$$
$$= \frac{\pi^3}{4} - \frac{\pi}{2}\left(\left[\frac{1}{2}x\sin 2x\right]_0^\pi - \frac{1}{2}\int_0^\pi \sin 2x \ \mathrm{d}x\right)$$
$$= \frac{\pi^3}{4} + \frac{\pi}{4}\left[-\frac{1}{2}\cos 2x\right]_0^\pi$$
$$= \frac{\pi^3}{4}$$
$$= 7.75 \text{ (GDC)}$$

19 **a**

$y = \sqrt{x^3 + 9}$
$y^2 = x^3 + 9$
$x^3 = y^2 - 9$
$x = \sqrt[3]{y^2 - 9}$
$x^2 = \sqrt[3]{(y^2 - 9)^2}$

b

$$A = \int_0^3 -x \ \mathrm{d}y$$
$$= \int_0^3 -(y^2 - 9)^{\frac{1}{3}} \ \mathrm{d}y$$
$$= 5.25 \text{ (GDC)}$$

ci

$$V_x = \pi \int_{-\sqrt[3]{9}}^0 y^2 \ \mathrm{d}x$$
$$= \pi \int_{-\sqrt[3]{9}}^0 x^3 + 9 \ \mathrm{d}x$$
$$= 44.1 \text{ (GDC)}$$

cii

$$V_y = \pi \int_0^3 x^2 \ \mathrm{d}y$$
$$= \pi \int_0^3 \sqrt[3]{(y^2 - 9)^2} \ \mathrm{d}x$$
$$= 30.1 \text{ (GDC)}$$

Worked solutions

20 Boundary points are $(1,1)$ and $\left(3, \frac{1}{3}\right)$, and the curve is $x = \frac{1}{y}$

$$V = \pi \int_{\frac{1}{3}}^{1} x^2 \, dy$$
$$= \pi \int_{\frac{1}{3}}^{1} y^{-2} \, dy$$
$$= \pi \left[-\frac{1}{y} \right]_{\frac{1}{3}}^{1}$$
$$= \pi(3 - 1)$$
$$= 2\pi$$

21

The volume if the entire rectangle with opposite vertices at the origin and at $(20, 4)$ is $\pi \times 20^2 \times 4$

The volume of revolution of the curve arc will give the inner volume V_y, so the required volume is the difference.

$y = \sqrt{x - 4}$ so $x = 4 + y^2$

$$V_y = \pi \int_0^4 x^2 \, dy$$
$$= \pi \int_0^4 16 + 8y^2 + y^4 \, dy$$
$$= \frac{6592}{15}\pi$$

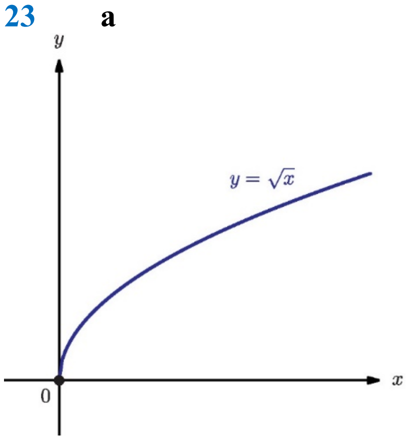
Then the required volume V is given by

$$V = \left(1600 - \frac{6592}{15}\right)\pi = \frac{17408}{15}\pi \approx 3646$$

22

$$V_x = \pi \int_0^{\pi} y^2 \, dx$$
$$= \pi \int_0^{\pi} \sin^2 x \, dx$$
$$= \frac{\pi^2}{2}$$

23 **a**



b

$$V_x = \pi \int_0^9 y^2 \, dx$$
$$= \pi \int_0^9 x \, dx$$
$$= \frac{81}{2}\pi \approx 127$$

c

$y = \sqrt{x}$ so $x = y^2$

The limits of the curve arc are $(0, 0)$ and $(9, 3)$

$$V_y = \pi \int_0^3 x^2 \, dy$$
$$= \pi \int_0^3 y^4 \, dy$$
$$= \frac{243}{5}\pi \approx 153$$

24 **a** $A: (0, 2), B: (2, 0)$

b

$$\text{Area} = \int_0^2 y \, dx$$
$$= \int_0^2 (2 - x)e^x \, dx$$
$$= 4.39 \text{ (GDC)}$$

c

$$V_x = \pi \int_0^2 y^2 \, dx$$
$$= \pi \int_0^2 (2 - x)^2 e^{2x} \, dx$$
$$= 32.7 \text{ (GDC)}$$

25

y-intercept is at $(0, 1)$

$y = \sqrt{x^2 + 1}$ so $x^2 = y^2 - 1$

$$V_y = \pi \int_1^3 x^2 \, dy$$
$$= \pi \int_1^3 y^2 - 1 \, dy$$
$$= \pi \left[\frac{1}{3}y^3 - y\right]_1^3$$
$$= \frac{20}{3}\pi$$

26

Volume of revolution of the shaded region will be given by the difference between the volume of revolution of the curve arc V_y and volume of revolution of the line (the latter producing a cylinder).

Intersection points are $(2, 0)$ and $(2, 2)$

$$V = \pi \int_0^2 x^2 \, dy - \pi \times 2^2 \times 2$$
$$= \pi \int_0^2 (2 + 2y - y^2)^2 \, dy - 8\pi$$
$$= \frac{72}{5}\pi - 8\pi \text{ (GDC)}$$
$$= \frac{32}{5}\pi$$

27 **a**

$$A = \int_0^{\ln a} x \, dy$$
$$= \int_0^{\ln a} e^y \, dy$$
$$= [e^y]_0^{\ln a}$$
$$= a - 1$$

b

The shaded area can also be seen as the area of the rectangle formed by the axes and with vertices at the origin and $(a, \ln a)$ less the unshaded area under the curve between $(1, 0)$ and $(a, \ln a)$.

$$a - 1 = a \ln a - \int_1^a y \, dx$$
$$= a \ln a - \int_1^a \ln x \, dx$$

So $\int_1^a \ln x \, dx = a \ln a - a + 1$

28 **a**

Gradient is $-\frac{h}{r}$, with intercept $(0, h)$

Line has equation $y = -\frac{h}{r}x + h$

$hx + ry = rh$

b

If the line between the two points is rotated about the y-axis, the resultant shape is a cone of radius r and height h, with axis along the y-axis.

$$V = \pi \int_0^h x^2 \, dy$$
$$= \pi \int_0^h \left(\frac{rh - ry}{h}\right)^2 dy$$
$$= \pi \int_0^h \frac{r^2}{h^2}(h^2 - 2hy + y^2) \, dy$$
$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$
$$= \frac{\pi r^2}{h^2}\left(h^3 - h^3 + \frac{1}{3}h^3\right)$$
$$= \frac{1}{3}\pi r^2 h$$

29 **a** $x^2 + y^2 = r^2$

b

Considering the curve $y = \sqrt{r^2 - x^2}$ for x between $-r$ and r, a sphere is formed by rotating the curve about the x-axis:

$$V = \pi \int_{-r}^{r} y^2 \, dx$$
$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$
$$= \pi \left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r}$$
$$= \pi\left(r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3\right)$$
$$= \frac{4}{3}\pi r^3$$

30 **a**

$y_1 = x^2, y_2 = \sqrt{x}$

Intersections where $y_1 = y_2$

$x^2 = \sqrt{x}$

$x^4 = x$

$x^3 = 1$ or 0

$x = 1$ or 0.

Intersections are at $(0, 0)$ and $(1, 1)$

b

$$V_x = \pi \int_0^1 y_2^2 - y_1^2 \, dx$$
$$= \pi \int_0^1 x - x^4 \, dx$$
$$= \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^4\right]_0^1$$
$$= \frac{3}{10}\pi$$

31

The region is symmetrical about $x = \frac{\pi}{4}$

so the calculation can be made by just considering the part of the region under $y = \sin x$ and doubling the result.

$$V = 2\pi \int_0^{\frac{\pi}{4}} \sin^2 x \, dx$$
$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2x) \, dx$$
$$= 2\pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= 2\pi\left(\frac{\pi}{8} - \frac{1}{4}\right)$$
$$= \frac{\pi^2 - 2\pi}{4}$$

32 **a**

$y_1 = x^2, y_2 = 2x$

Intersections where $y_1 = y_2$

$x^2 = 2x$

$x = 0$ or 2

Intersections are $(0, 0)$ and $(2, 4)$

b

$$V_y = \pi \int_0^4 x_1^2 - x_2^2 \, dy$$
$$= \pi \int_0^4 y - \left(\frac{y}{2}\right)^2 dy$$
$$= \frac{8}{3}\pi$$

33

The boundary points on the curve are at (p, p^3) and (q, q^3)

$$A = \int_p^q y \, dx = \left[\frac{1}{4}x^4\right]_p^q = \frac{1}{4}(q^4 - p^4)$$

$$B = \int_{p^3}^{q^3} x \, dy = \int_{p^3}^{q^3} y^{\frac{1}{3}} \, dy = \left[\frac{3}{4}y^{\frac{4}{3}}\right]_{p^3}^{q^3} = \frac{3}{4}(q^4 - p^4)$$

The ratio of A: B is therefore 1: 3, independent of p and q.

34 **a** $y = \ln(x-2)$

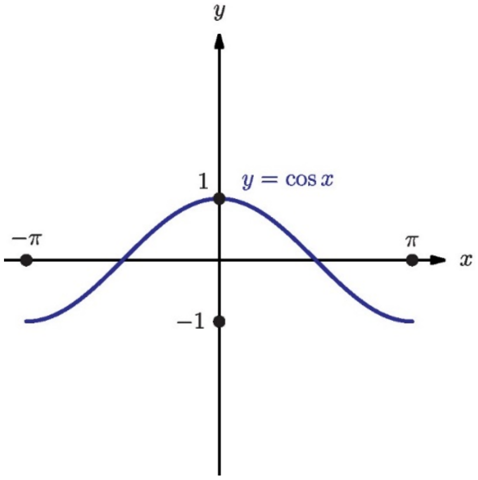
b

$x = e^y + 2$

The volume obtained rotating $y = \ln x$ for $1 \le x \le e$ is the same as when the curve $y = \ln(x-2)$ for $3 \le x \le e+2$ is rotated about the line $x = 0$. Boundary points are $(3, 0)$ and $(e+2, 1)$

$$V_y = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 e^{2y} + 4e^y + 4 \, dy$$
$$= \pi \left[\frac{1}{2}e^{2y} + 4e^y + 4y\right]_0^1$$
$$= \pi \left(\frac{1}{2}e^2 + 4e + 4 - \frac{1}{2} - 4\right)$$
$$= \frac{\pi}{2}(e^2 + 8e - 1)$$

35 **a**



b

The volume obtained is the same as for $y = \cos x + 1$ rotated about the line $y = 0$ (the x-axis).

$$V_x = \pi \int_{-\pi}^{\pi} y^2 \, dx$$
$$= \pi \int_{-\pi}^{\pi} \cos^2 x + 2\cos x + 1 \, dx$$
$$= \pi \int_{-\pi}^{\pi} \frac{1}{2}(\cos 2x + 1) + 2\cos x + 1 \, dx$$
$$= \pi \left[\frac{1}{4}\sin 2x + 2\sin x + \frac{3}{2}x\right]_{-\pi}^{\pi}$$
$$= 3\pi^2$$

36 **a**

$y = \arccos x$ so $x = \cos y$

$$R = \int_{\theta}^{\frac{\pi}{2}} x \, \mathrm{d}y$$

$$= [\sin y]_{\theta}^{\frac{\pi}{2}}$$

$$= 1 - \sin\theta$$

b

When $x = a, y = \theta$ so $a = \cos\theta$

Then $\sin\theta = \sqrt{1-a^2}$

c

The area comprising both R and the rectangle below it is given by the integral

$$\int_0^a y \, \mathrm{d}x$$

The rectangle has area $a\theta = a \arccos a$

Therefore

$$\int_0^a \arccos x \, \mathrm{d}x = R + a \arccos a = 1 - \sqrt{1-a^2} + a \arccos a$$

Mixed Practice

1

$y = \arcsin(3x)$

$$y' = \frac{3}{\sqrt{1-9x^2}} \text{ so } y'\left(\frac{1}{6}\right) = \frac{3}{\sqrt{\frac{3}{4}}} = 2\sqrt{3}$$

2

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sec^2(2x) \, \mathrm{d}x = \left[\frac{1}{2}\tan(2x)\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)$$

$$= \frac{\sqrt{3}}{3}$$

3 **a**

$y = \sec x$

$y' = \sec x \tan x$

$y'' = \sec x \tan^2 x + \sec^3 x$

b

A point of inflection occurs when $y''(x) = 0$

$y''(x) = \sec^3 x \,(\sin^2 x + 1)$

Since $\sec x \neq 0$ for any real x and $\sin^2 x + 1 \geq 1$ for real x, it follows that $y''(x) \neq 0$ for real x.

Therefore there are no points of inflection.

4

$y = 10^x$

$y' = 10^x \ln 10$

$y'' = 10^x (\ln 10)^2$

$y''' = 10^x (\ln 10)^3$

5

$$V = \pi \int_1^{2e} y^2 \, dx$$

$$= \pi \int_1^{2e} (\ln x)^2 \, dx$$

$$= 19.0 \text{ (GDC)}$$

6

Implicit differentiation:

$6x^2 - 15y^2y' = 0$

$$y' = \frac{2x^2}{5y^2}$$

$$y'|_{(2,1)} = \frac{8}{5}$$

7

Let $3x = u$ so $3\,dx = du$

$$\int \frac{6}{1+9x^2} \, dx = \int \frac{2}{1+u^2} \, du$$

$$= 2 \arctan u + c$$

$$= 2 \arctan(3x) + c$$

8

If a side length is x then Volume $V = x^3$

$\dot{V} = 3x^2 \dot{x}$

When $x = 12, \dot{x} = 0.6$ then $\dot{V} = 259.2 \text{ cm}^3 \text{ s}^{-1}$

9

$y = 3\sin(2\pi x)$

$y' = 6\pi \cos(2\pi x)$

$$y'\left(\frac{7}{12}\right) = 6\pi \times \left(-\frac{\sqrt{3}}{2}\right) = -3\pi\sqrt{3} \approx -16.3$$

10

Let $y = x^2 e^{-x}$ for $-3 \le x \le 3$

$y' = (2x - x^2)e^{-x} = x(2-x)e^{-x}$

$y' = 0 \Rightarrow x = 0$ or $x = 2$

$y(-3) = 9e^3 \approx 181$

$y(0) = 0$

$y(2) = 4e^{-2} \approx 0.541$

$y(3) = 9e^{-3} \approx 0.448$

The maximum value is approximately 181

11 **a**

Surface area equals the two square $a \times a$ faces and the four rectangular $a \times h$ faces.

$S = 2a^2 + 4ah$

$V = a^2 h = 1000$ so $h = 1000a^{-2}$

Then $S = 2a^2 + 4000a^{-1}$

b

$S' = 4a - 4000a^{-2}$

$S' = 0 \Rightarrow a = 1000a^{-2}$

$a^3 = 1000$

$a = 10$

c)

$S'' = 4 + 8000a^3 > 0$ for $a = 10$ so this represents a minimum of the function S

$S(10) = 200 + 400 = 600\text{ cm}^2$

12 **a**

$\frac{5-x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$ for some constants A and B

Multiplying by the denominator on the LHS:

$5 - x = A(x-2) + B(x+1)$

Substituting:

$x = -1: 6 = -3A \Rightarrow A = -2$

$x = 2: 3 = 3B \Rightarrow B = 1$

$\frac{5-x}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{2}{x+1}$

b)

$$\begin{aligned}\int_3^5 \frac{5-x}{x^2-x-2}\,\mathrm{d}x &= \int_3^5 \frac{1}{x-2} - \frac{2}{x+1}\,\mathrm{d}x \\ &= [\ln|x-2| - 2\ln|x+1|]_3^5 \\ &= (\ln 3 - 2\ln 6) - (\ln 1 - 2\ln 4) \\ &= \ln 3 - 2\ln 6 + 2\ln 4 \\ &= \ln\left(\frac{3\times 4^2}{6^2}\right) \\ &= \ln\left(\frac{4}{3}\right)\end{aligned}$$

13

Implicit differentiation:

$3x^2y^3 - y + (3x^3y^2 - x)y' = 0$

$y' = \frac{y - 3x^2y^3}{3x^3y^2 - x}$

$y'|_{(1,1)} = -\frac{2}{2} = -1$

Normal gradient is therefore 1, through $(1, 1)$.

Normal has equation $y = x$

14 **a**

$y = xe^{3x}$

$y' = (1+3x)e^{3x}$

b

<u>Proposition:</u> $y^{(n)}(x) = (n3^{n-1} + x3^n)e^{3x}$ for $n \geq 1$

<u>Base case:</u> $y'(x) = (1+3x)e^{3x} = (1\times 3^0 + x\times 3^1)e^{3x}$ so the proposition is true for $n = 1$

<u>Inductive step:</u> Assume the proposition is true for $n = k \geq 1$

So $y^{(k)}(x) = (k3^{k-1} + x3^k)e^{3x}$

<u>Working towards:</u> $y^{(k+1)}(x) = \left((k+1)3^k + x3^{k+1}\right)e^{3x}$

$$y^{(k+1)}(x) = \frac{d}{dx}y^{(k)}(x)$$
$$= \frac{d}{dx}(k3^{k-1} + x3^k)e^{3x} \text{ using the assumption}$$
$$= (3[k3^{k-1} + x3^k] + 3^k)e^{3x}$$
$$= (k3^k + x3^{k+1} + 3^k)e^{3x}$$
$$= \left((k+1)3^k + x3^{k+1}\right)e^{3x}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

c

Stationary points occur wherever $y' = 0$

$(1 + 3x)e^{3x} = 0$ has a single solution at $x = -\frac{1}{3}$

$$y''\left(-\frac{1}{3}\right) = \left(6 + 9\left(-\frac{1}{3}\right)\right)e^{-1}$$

> 0 so this is a local minimum and not a point of inflection

$y\left(-\frac{1}{3}\right) = -\frac{1}{3e}$ so the minimum is at $\left(-\frac{1}{3}, -\frac{1}{3e}\right)$

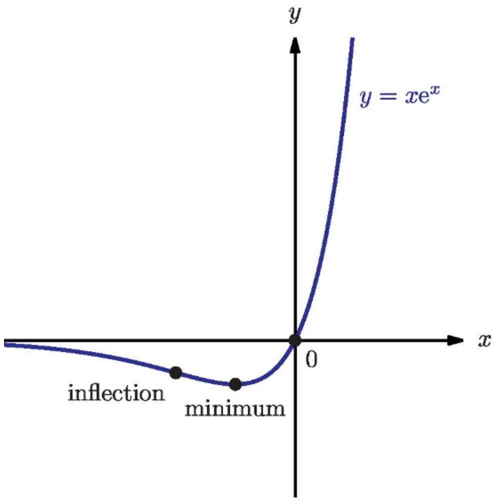
d

Points of inflection occur when $y'' = 0$

$(6 + 9x)e^{3x} = 0$ has a single solution at $x = -\frac{2}{3}$

$y\left(-\frac{2}{3}\right) = -\frac{2}{3e^2}$ so the minimum is at $\left(-\frac{2}{3}, -\frac{2}{3e^2}\right)$

e



15

First principles differentiation: $\frac{dy}{dx} = \lim_{h\to 0}\left\{\frac{y(x+h)-y(x)}{h}\right\}$

$y = x^3 - 3x$

$$\frac{dy}{dx} = \lim_{h\to 0}\left\{\frac{(x+h)^3 - 3(x+h) - (x^3-3x)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x^3+3x^2h+3xh^2+h^3-3x-3h-x^3+3x}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{3x^2h+3xh^2+h^3-3h}{h}\right\}$$
$$= \lim_{h\to 0}\{3x^2+2xh+h^2-3\}$$
$$= 3x^2-3$$

16

$y = e^{2x+1}$

Proposition: $y^{(n)}(x) = 2^n e^{2x+1}$ for $n \geq 0$

Base case: $y^{(0)}(x) = y(x) = 2^0 e^{2x+1}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $y^{(k)}(x) = 2^k e^{2x+1}$

Working towards: $y^{(k+1)}(x) = 2^{k+1}e^{2x+1}$

$$y^{(k+1)}(x) = \frac{d}{dx}y^{(k)}(x)$$
$$= \frac{d}{dx}2^k e^{2x+1} \text{ using the assumption}$$
$$= 2\times 2^k e^{2x+1}$$
$$= 2^{k+1}e^{2x+1}$$

So the proposition is true for $n = k+1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

17

$y = y = \frac{1}{1-2x} = (1-2x)^{-1}$

Proposition: $y^{(n)}(x) = 2^n n!\,(1-2x)^{-n-1}$ for $n \geq 0$

Base case: $y^{(0)}(x) = y(x) = 2^0 \times 0! \times (1-2x)^{-0-1}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $y^{(k)}(x) = 2^k k!\,(1-2x)^{-k-1}$

Working towards: $y^{(k+1)}(x) = 2^{k+1}(k+1)!\,(1-2x)^{-k-2}$

$$y^{(k+1)}(x) = \frac{d}{dx}y^{(k)}(x)$$
$$= \frac{d}{dx}2^k k!\,(1-2x)^{-k-1} \text{ using the assumption}$$
$$= -2(-k-1)\times 2^k k!\,(1-2x)^{-k-2} \text{ using the Chain Rule}$$
$$= 2^{k+1}(k+1)!\,(1-2x)^{-k-2}$$

So the proposition is true for $n = k+1$

Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

18

$y = x^2 e^x$

Proposition: $y^{(n)}(x) = (x^2 + 2nx + n(n-1))e^x$ for $n \geq 0$

Base case: $y^{(0)}(x) = y(x) = \left(x^2 + 2(0)x + 0(0-1)\right)e^x$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for $n = k \geq 0$

So $y^{(k)}(x) = \left(x^2 + 2kx + k(k-1)\right)e^x$

Working towards: $y^{(k+1)}(x) = (x^2 + 2(k+1)x + (k+1)k)e^x$

$$y^{(k+1)}(x) = \frac{d}{dx} y^{(k)}(x)$$
$$= \frac{d}{dx}\left(x^2 + 2kx + k(k-1)\right)e^x \text{ using the assumption}$$
$$= (x^2 + 2kx + k(k-1) + 2x + 2k)e^x \text{ using the Product Rule}$$
$$= \left(x^2 + 2(k+1)x + k(k+1)\right)e^x$$

So the proposition is true for $n = k + 1$

Conclusion:
The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

19 **a**

$f(2) = 2^2 = 4 = k \times 3^2$

$$k = \frac{4}{9}$$

b

$$f'(x) = \begin{cases} 2^x \ln 2 & (x < 2) \\ \frac{4}{9} \times 3^x \ln 3 & (x > 2) \end{cases}$$

Since $f'(x)$ is not continuous at $x = 2$, $f(x)$ is not differentiable here.

20

Using L'Hôpital's rule:

$$\lim_{x \to 0}\left\{\frac{3^x - 1}{x}\right\} = \lim_{x \to 0}\left\{\frac{3^x \ln 3}{1}\right\} = \ln 3$$

21

Using L'Hôpital's rule (repeatedly):

$$\lim_{x \to 0}\left\{\frac{\sin x - x}{\tan x - x}\right\} = \lim_{x \to 0}\left\{\frac{\cos x - 1}{\sec^2 x - 1}\right\}$$
$$= \lim_{x \to 0}\left\{\frac{-\sin x}{2\sec^2 x \tan x}\right\}$$
$$= \lim_{x \to 0}\left\{\frac{-\cos x}{2\sec^4 x + 2\sec^2 \tan^2 x}\right\}$$
$$= -\frac{1}{2}$$

22

$$\int_0^2 \frac{2x+5}{x^2+4}\,dx = \int_0^2 \frac{2x}{x^2+4} + \frac{5}{x^2+4}\,dx$$
$$= \left[\ln(x^2+4) + \frac{5}{2}\arctan\left(\frac{x}{2}\right)\right]_0^2$$
$$= \left(\ln 8 + \frac{5}{2}\left(\frac{\pi}{4}\right)\right) - (\ln 4 + 0)$$
$$= \ln 2 + \frac{5\pi}{8}$$

23 **a** $a = \ln 11$

b

$y = \ln(5x+1)$ so $x = \frac{1}{5}(e^y - 1)$

$$\text{Area} = \int_0^{\ln 11} x\,dy$$
$$= \int_0^{\ln 11} \frac{1}{5}e^y - \frac{1}{5}\,dy$$
$$= \left[\frac{1}{5}e^y - \frac{1}{5}y\right]_0^{\ln 11}$$
$$= \frac{11}{5} - \frac{1}{5}\ln 11 - \frac{1}{5}$$
$$= 2 - \frac{1}{5}\ln 11 \approx 1.52 \text{ (GDC)}$$

c

$$\text{Volume} = \pi \int_0^{\ln 11} x^2\,dy$$
$$= \pi \int_0^{\ln 11} \frac{1}{25}(e^{2y} - 2e^y + 1)\,dy$$
$$= \pi\left[\frac{1}{50}e^{2y} - \frac{2}{25}e^y + \frac{1}{25}y\right]_0^{\ln 11}$$
$$= \pi\left(\frac{121}{50} - \frac{22}{25} + \frac{1}{25}\ln 11\right) - \pi\left(\frac{1}{50} - \frac{2}{25}\right)$$
$$= \frac{(40 + \ln 11)\pi}{25} \approx 5.33 \text{ (GDC)}$$

24 **a**

If A has coordinates $(a, 0)$ then the base of the rectangle has length $2a$ and the height is $9 - a^2$

Area $= 2a(9 - a^2)$

$$\frac{d\,\text{Area}}{da} = 18 - 6a^2$$

Stationary value when $a = \sqrt{3}$

A has coordinates $(\sqrt{3}, 0)$

b If $a = 0$ or 3, the area is zero.

25

$y = \ln(x^2)$ so $y(1) = 0, y(e^2) = 4$

$x^2 = \mathrm{e}^y$

$$\text{Volume} = \pi \int_0^4 x^2 \ \mathrm{d}y$$
$$= \pi \int_0^4 \mathrm{e}^y \ \mathrm{d}y$$
$$= \pi[\mathrm{e}^y]_0^4$$
$$= \pi(\mathrm{e}^4 - 1)$$

26

$y = \cos x$

$$\text{Volume} = \pi \int_0^{\frac{\pi}{2}} y^2 \ \mathrm{d}x$$
$$= \pi \int_0^{\frac{\pi}{2}} \cos^2 x \ \mathrm{d}x$$
$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \ \mathrm{d}x$$
$$= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi^2}{4}$$

27

Curve intersection at $(1, 1)$

First curve: $y_1 = |x|$ so $x^2 = y_1^2$

Second curve: $y_2 = 2 - x^4$ so $x^2 = \sqrt{2 - y_2}$

$$\text{Volume} = \pi \int_0^1 x^2 \ \mathrm{d}y_1 + \pi \int_1^2 x^2 \ \mathrm{d}y_2$$
$$= \pi \left(\int_0^1 y_1^2 \ \mathrm{d}y_1 + \int_1^2 (2 - y_2)^{0.5} \ \mathrm{d}y_2 \right)$$
$$= \pi \left(\left[\frac{1}{3}y^3\right]_0^1 + \left[-\frac{2}{3}(2 - y)^{1.5}\right]_1^2 \right)$$
$$= \pi \left(\frac{1}{3} + \frac{2}{3} \right)$$
$$= \pi$$

28

Axis intercepts are $(\pm 3, 0)$ and $(0, \pm 2)$

Volume rotated about x-axis:

$$V_x = \pi \int_{-2}^2 y^2 \ \mathrm{d}x$$
$$= \pi \int_{-2}^2 \left(4 - \frac{4}{9}x^2\right) \ \mathrm{d}x$$
$$= \pi \left[4x - \frac{4}{27}x^3\right]_{-2}^2$$

$$= \pi\left(16 - \frac{64}{27}\right)$$
$$= \frac{368}{27}\pi$$

Volume rotated about y-axis:

$$V_y = \pi\int_{-3}^{3} x^2 \, dy$$
$$= \pi\int_{-3}^{3}\left(9 - \frac{9}{4}y^2\right) dy$$
$$= \pi\left[9y - \frac{1}{4}y^3\right]_{-3}^{3}$$
$$= \pi\left(54 - \frac{54}{4}\right)$$
$$= \frac{81}{2}\pi$$

29

Let $u = \ln x$, $v' = x^3$ so $u' = x^{-1}$, $v = \frac{1}{4}x^4$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$$\int_1^{e} x^3 \ln x \, dx = \left[\frac{1}{4}x^4 \ln x\right]_1^{e} - \int_1^{e} \frac{1}{4}x^3 \, dx$$
$$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^{e}$$
$$= \frac{1}{16}(3e^4 + 1)$$

30

Let $u = \sqrt{x+1}$ so $x = u^2 - 1$ and then $dx = 2u\, du$

Limits:

When $x = -1, u = 0$

When $x = 3, u = 2$

$$\int_{-1}^{3} \frac{1}{2}e^{\sqrt{x+1}} \, dx = \int_0^2 \frac{1}{2}e^u \times 2u \, du$$
$$= \int_0^2 ue^u \, du$$
$$= [ue^u]_0^2 - \int_0^2 e^u \, du \text{(using integration by parts)}$$
$$= [ue^u - e^u]_0^2$$
$$= e^2 + 1$$

31 **a**

Let $u = t, \dot{v} = e^{-t}$ so $\dot{u} = 1, v = -e^{-t}$

Integration by parts: $\int u\dot{v} \, dt = uv - \int \dot{u}v \, dt$

$$\int te^{-t} \, dt = -te^{-t} + \int e^{-t} \, dt$$
$$= -(t+1)e^{-t} + c$$

Worked solutions

b

Let $u = x^2$ so $\mathrm{d}u = 2x\ \mathrm{d}x$

Limits:

When $x = 0, u = 0$

When $x = 1, u = 1$

$$\int_0^1 2x^3\mathrm{e}^{-x^2}\ \mathrm{d}x = \int_0^1 x^2\mathrm{e}^{-x^2} \times 2x\ \mathrm{d}x$$
$$= \int_0^1 u\mathrm{e}^{-u}\ \mathrm{d}u$$
$$= [-(u+1)\mathrm{e}^{-u}]_0^1$$
$$= -(2\mathrm{e}^{-1}) + 1$$
$$= 1 - \frac{2}{\mathrm{e}}$$

32

Let $u = x^2 + 1$ so $\mathrm{d}u = 2x\ \mathrm{d}x$

$$\int x^3\sqrt{x^2+1}\ \mathrm{d}x = \int \frac{1}{2}x^2\sqrt{x^2+1} \times 2x\ \mathrm{d}x$$
$$= \int \frac{1}{2}(u-1)\sqrt{u}\ \mathrm{d}u$$
$$= \int \frac{1}{2}u^{1.5} - \frac{1}{2}u^{0.5}\ \mathrm{d}u$$
$$= \frac{1}{5}u^{2.5} - \frac{1}{3}u^{1.5} + c$$
$$= \frac{1}{5}(x^2+1)^{2.5} - \frac{1}{3}(x^2+1)^{1.5} + c$$

33 **a**

Let $u = \sqrt{x}$ so $x = u^2$ and so $dx = 2u\ du$

$$\int \sqrt{x}\mathrm{e}^{\sqrt{x}}\ \mathrm{d}x = \int u\mathrm{e}^u \times 2u\ \mathrm{d}u$$
$$= \int 2u^2\mathrm{e}^u\ \mathrm{d}u$$

b

Limits:

When $x = 0, u = 0$

When $x = 4, u = 2$

Let $p = u^2, \frac{\mathrm{d}^2q}{\mathrm{d}u^2} = \mathrm{e}^u$ so $\frac{\mathrm{d}p}{\mathrm{d}u} = 2u, \frac{\mathrm{d}q}{\mathrm{d}u} = \mathrm{e}^u, \frac{\mathrm{d}^2p}{\mathrm{d}u^2} = 2, q = \mathrm{e}^u$

Integration by parts (twice): $\int p\frac{\mathrm{d}q^2}{\mathrm{d}u^2}\mathrm{d}u = p\frac{\mathrm{d}q}{\mathrm{d}u} - \int \frac{\mathrm{d}p}{\mathrm{d}u}\frac{\mathrm{d}q}{\mathrm{d}u}\ \mathrm{d}u$

$$= p\frac{\mathrm{d}q}{\mathrm{d}u} - \frac{\mathrm{d}p}{\mathrm{d}u}q + \int \frac{\mathrm{d}^2p}{\mathrm{d}u^2}q\ \mathrm{d}u$$

$$\int_0^2 2u^2\mathrm{e}^u\ \mathrm{d}u = [\mathrm{e}^u(u^2-2u)]_0^2 + \int_0^2 2\mathrm{e}^u\ \mathrm{d}u$$
$$= [\mathrm{e}^u(u^2-2u+2)]_0^2$$
$$= 4\mathrm{e}^2 - 4 + 2 - 2$$
$$= 4(\mathrm{e}^2 - 1)$$

34

Let $x = \tan\theta$ so $dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2\theta}\sec^2\theta\; d\theta$$
$$= \int 1\; d\theta$$
$$= \theta + c$$
$$= \arctan x + c$$

35

$$f(x) = \begin{cases} e^{-x^2}(-x^3+2x^2+x) & x \le 1 \\ ax+b & x > 1 \end{cases}$$

Then

$$f'(x) = \begin{cases} e^{-x^2}(-3x^2+4x+1+2x^4-4x^3-2x^2) & x \le 1 \\ a & x > 1 \end{cases}$$

If the function is continuous at $x = 1$ then $2e^{-1} = a + b$

If the function is differentiable at $x = 1$ then $e^{-1}(-3+4+1+2-4-2) = -2e^{-1} = a$

$b = 2e^{-1} - a = 4e^{-1}$

36

$y_1 = \sqrt{x}\,e^x$ and $y_2 = e\sqrt{x}$ Intersections occur when $\sqrt{x}e^x = e\sqrt{x}$

$\sqrt{x}(e^x - e) = 0$

$x = 0$ or $x = 1$

$$\text{Volume} = \pi\int_0^1 y_2^2 - y_1^2\; dx$$
$$= \pi\int_0^1 xe^2 - xe^{2x}\; dx$$
$$= \pi\left[\frac{1}{2}x^2e^2\right]_0^1 - \pi\int_0^1 xe^{2x}\; dx$$
$$= \frac{1}{2}\pi e^2 - \pi\int_0^1 xe^{2x}\; dx$$

Let $u = x, v' = e^{2x}$ so $u' = 1, v = \frac{1}{2}e^{2x}$

Integration by parts: $\int uv'\,dx = uv - \int u'v\; dx$

$$\pi\int_0^1 xe^{2x}\; dx = \pi\left(\left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\; dx\right)$$
$$= \pi\left[\left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}\right]_0^1$$
$$= \pi\left(\frac{1}{4}e^2 + \frac{1}{4}\right)$$

$$\text{Volume} = \pi\left(\frac{1}{2}e^2 - \left(\frac{1}{4}e^2 + \frac{1}{4}\right)\right)$$
$$= \frac{\pi}{4}(e^2 - 1)$$

37 **a**

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

b

Let $u = \arcsin x$, $v' = 1$ so $u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$

Integration by parts: $\int uv'\,dx = uv - \int u'v\,dx$

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= x\arcsin x + \sqrt{1-x^2} + c$$

38 **a**

$$x^2 - 8x + 25 = (x-4)^2 - 16 + 25$$
$$= (x-4)^2 + 9$$

b

$$\int \frac{2x+7}{x^2-8x+25}\,dx = \int \frac{2x-8+15}{x^2-8x+25}\,dx$$
$$= \int \frac{2x-8}{x^2-8x+25} + \frac{15}{(x-4)^2+9}\,dx$$
$$= \ln|x^2-8x+25| + \frac{15}{3}\arctan\left(\frac{x-4}{3}\right) + c$$
$$= \ln(x^2-8x+25) + 5\arctan\left(\frac{x-4}{3}\right) + c$$

It is not incorrect to write the logarithm argument as a modulus, but the completed square shows that it is always positive, so this is not actually necessary in this case.

39

Let one of the vertices be at $(a, 0)$.

Then the base of the rectangle is $2a$ and the height of the rectangle is $\sqrt{r^2-a^2}$

Area $= 2a\sqrt{r^2-a^2} = 2a(r^2-a^2)^{0.5}$

$$\frac{d\,\text{Area}}{da} = 2(r^2-a^2)^{0.5} - 2a^2(r^2-a^2)^{-0.5}$$
$$= (2r^2 - 2a^2 - 2a^2)(r^2-a^2)^{-0.5}$$

Stationary point, which represents maximum area (since minimum area is clearly at $a = 0$ or $a = r$):

$$2r^2 - 4a^2 = 0 \Rightarrow a = \frac{r}{\sqrt{2}}$$

$$\text{Area}\left(\frac{r}{\sqrt{2}}\right) = \frac{2r}{\sqrt{2}} \times \frac{r}{\sqrt{2}} = r^2$$

40 **a** Let $x = a\sin\theta$ so $dx = a\cos\theta\ d\theta$

$$\int \sqrt{a^2 - x^2}\ dx = \int a\cos\theta \times a\cos\theta\ d\theta$$

$$= a^2 \int \cos^2\theta\ d\theta$$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\ d\theta$$

$$= \frac{a^2}{4}(2\theta + \sin 2\theta) + c$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + c$$

$$= \frac{a^2}{2}(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\left(\sqrt{1 - \left(\frac{x}{a}\right)^2}\right) + c$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + c$$

b

$$\text{Area} = \int_{-a}^{-a} y\ dx$$

$$= \int_{-a}^{a} \sqrt{a^2 - x^2}\ dx$$

$$= \left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_{-a}^{a}$$

$$= \left(\frac{a^2}{2}\left(\frac{\pi}{2}\right) + 0\right) - \left(\frac{a^2}{2}\left(-\frac{\pi}{2}\right) + 0\right)$$

$$= \frac{a^2\pi}{2}$$

Radius of the semicircle is $a = 6$, so Area $= 18\pi$

41

Let $x = \frac{e^u + e^{-u}}{2}$ so $dx = \frac{e^u - e^{-u}}{2}\ du$

$e^u + e^{-u} = 2x$ so $e^{2u} - 2xe^u + 1 = 0, e^u$

$= \left(x \pm \sqrt{x^2 - 1}\right)$; take the positive root wlog.

$$x^2 = \frac{e^{2u} + 2 + e^{-2u}}{4} \text{ so } x^2 - 1 = \frac{e^{2u} - 2 + e^{-2u}}{4} = \left(\frac{e^u - e^{-u}}{2}\right)^2$$

Limits:

When $x = 2, u = \ln\left(2 + \sqrt{3}\right)$

When $x = 4, u = \ln\left(4 + \sqrt{15}\right)$

$$\int_2^4 \frac{1}{\sqrt{x^2 - 1}}\ dx = \int_{\ln(2+\sqrt{3})}^{\ln(4+\sqrt{15})} 1\ du$$

$$= [u]_{\ln(2+\sqrt{3})}^{\ln(4+\sqrt{15})}$$

$$= \ln\left(4 + \sqrt{15}\right) - \ln\left(2 + \sqrt{3}\right)$$

$$= \ln\left(\frac{4 + \sqrt{15}}{2 + \sqrt{3}}\right)$$

42

Let r be the distance from the bird to the observer (obliquely)

Then $r = 35\operatorname{cosec}\theta$

Implicit differentiation:

$\dot{r} = -35\operatorname{cosec}\theta\cot\theta\,\dot{\theta}$

When $\theta = 1.2, \dot{\theta} = \dfrac{1}{60}$

$\dot{r} = -\dfrac{35\operatorname{cosec}1.2\cot 1.2}{60} = -0.243\text{ m s}^{-1}$

The bird's distance to the observer is decreasing at a rate 0.243 m s^{-1}

43

Let d be the distance from the point $(0, 1)$ to a point on the curve $(x, \sin x)$

$d = \sqrt{x^2 + (\sin x - 1)^2}$

From GDC, minimum d is $d(0.478) = 0.721$

44

Using L'Hôpital's Rule:

$$\lim_{x\to\frac{\pi}{2}}\{\tan x - \sec x\} = \lim_{x\to\frac{\pi}{2}}\left\{\frac{\sin x - 1}{\cos x}\right\}$$
$$= \lim_{x\to\frac{\pi}{2}}\left\{\frac{\cos x}{-\sin x}\right\}$$
$$= 0$$

45

Let $L = \lim\limits_{x\to\infty}\left\{x\arctan x - \dfrac{\pi x}{2}\right\}$

$$L = \lim_{x\to\infty}\left\{\frac{x^2\left(\arctan x - \frac{\pi}{2}\right)}{x}\right\}$$
$$= \lim_{x\to\infty}\left\{2x\left(\arctan x - \frac{\pi}{2}\right) + \frac{x^2}{1+x^2}\right\} \text{ using L'Hôpital's Rule}$$
$$= \lim_{x\to\infty}\left\{x(2\arctan x - \pi) + 1 - \frac{1}{1+x^2}\right\}$$
$$= \lim_{x\to\infty}\left\{2x\left(\arctan x - \frac{\pi}{2}\right)\right\} + 1$$
$$= 2L + 1$$

Rearranging: $L = -1$

46 **a**

Using compound angle formula:

$$\cos\left(x + \frac{\pi}{2}\right) = \cos x\cos\left(\frac{\pi}{2}\right) - \sin x\sin\left(\frac{\pi}{2}\right)$$
$$= 0\cos x - 1\sin x$$
$$= -\sin x$$

b

Proposition: $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(\cos x) = \cos\left(x + \dfrac{n\pi}{2}\right)$ for $n \geq 1$

Base case: $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x = \cos\left(x + \dfrac{\pi}{2}\right)$ from part **a** so the proposition is true for $n = 1$

Inductive step: Assume the proposition is true for $n = k \geq 1$

So $\dfrac{\mathrm{d}^k}{\mathrm{d}x^k}(\cos x) = \cos\left(x + \dfrac{k\pi}{2}\right)$

<u>*Working towards:*</u> $\frac{d^{k+1}}{dx^{k+1}}(\cos x) = \cos(x + \frac{(k+1)\pi}{2})$

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(\cos x) &= \frac{d}{dx}\left(\frac{d^k}{dx^k}(\cos x)\right)\\ &= \frac{d}{dx}\left(\cos\left(x + \frac{k\pi}{2}\right)\right) \text{ using the assumption}\\ &= -\sin\left(x + \frac{k\pi}{2}\right)\\ &= \cos\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right) \text{ using part } \mathbf{a}\\ &= \cos\left(x + \frac{(k+1)\pi}{2}\right)\end{aligned}$$

So the proposition is true for $n = k + 1$

<u>Conclusion</u>:

The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

47

$e^x + ye^{-x} = 2e^3$

Implicit differentiation:

$e^x - e^{-x}y + e^{-x}y' = 0$

$y' = y - e^{2x}$

Stationary point occurs when $y' = 0$

$y = e^{2x}$

Substituting into the curve equation, $y = 2e^{3+x} - e^{2x}$:

$2e^{2x} = 2e^{x+3}$ so $x = 3$

The stationary point is at $(3, e^6)$

48 **a**

$x^2 + xy + y^2 = 12$

Implicit differentiation:

$2x + y + (x + 2y)y' = 0$

$$y' = -\frac{2x + y}{x + 2y}$$

Stationary points occur where $y' = 0$:

$y = -2x$

Substituting into the curve equation:

$3x^2 = 12$

$x = \pm 2$

Stationary points are $(2, -4)$ and $(-2, 4)$

b

$2x + y + (x + 2y)y' = 0$

Implicit differentiation:

$2 + y' + (1 + 2y')y' + (x + 2y)y'' = 0$

$(x + 2y)y'' = -2 - (2 + 2y')y'$

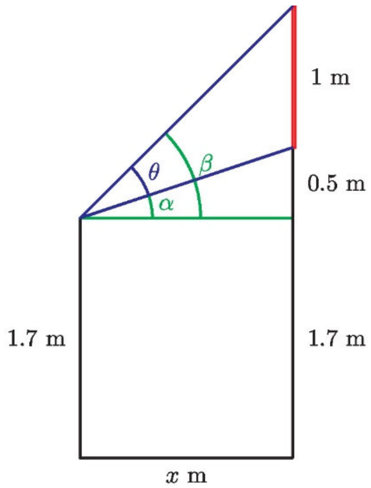
At stationary points, $y' = 0$, so $(x + 2y)y'' = -2$

c

At $(2, -4), -6y'' = -2$ so $y'' > 0$: $(2, -4)$ is a local minimum

At $(-2, 4), 6y'' = -2$ so $y'' > 0$: $(-2,4)$ is a local maximum

49 **a**



In the diagram above:

$\alpha = \arctan\left(\frac{0.5}{x}\right), \beta = \arctan\left(\frac{1.5}{x}\right)$

$\theta = \beta - \alpha = \arctan\left(\frac{1.5}{x}\right) - \arctan\left(\frac{0.5}{x}\right)$

b

$$\theta' = -\frac{1.5}{x^2}\left(\frac{1}{1+\left(\frac{1.5}{x}\right)^2}\right) + \frac{0.5}{x^2}\left(\frac{1}{1+\left(\frac{0.5}{x^2}\right)^2}\right)$$

$$= \frac{0.5}{x^2+0.25} - \frac{1.5}{x^2+2.25}$$

$$= \frac{0.5(x^2+2.25) - 1.5(x^2+0.25)}{(x^2+0.25)(x^2+2.25)}$$

$$= \frac{-x^2+0.75}{(x^2+0.25)(x^2+2.25)}$$

Maximum θ occurs when $\theta' = 0$:

$x = \frac{\sqrt{3}}{2}$ (reject negative root, given context)

The picture will appear as large as possible when $x = 0.5\sqrt{3} \approx 0.866$ m

At this distance, $\theta = \arctan(\sqrt{3}) - \arctan\left(\frac{1}{\sqrt{3}}\right)$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

50 **a**

Gradient is $\frac{h}{b-a}$, passing through $(a, 0)$

Line equation is $y = \frac{h}{b-a}(x-a)$

b

Rearranging the line equation:

$$x = a + \frac{(b-a)y}{h}$$

$$V = \pi\int_0^h x^2\ dy$$

$$= \pi\int_0^h \frac{(ah+(b-a)y)^2}{h^2}\ \mathrm{d}y$$

$$= \frac{\pi}{h^2}\int_0^h a^2h^2 + 2a(b-a)y + (b-a)^2y^2\ \mathrm{d}y$$

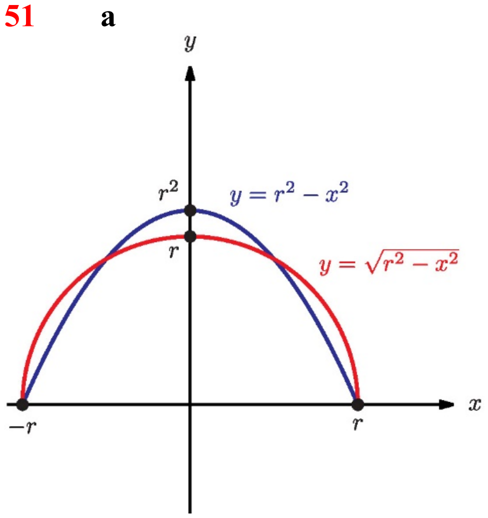
$$= \left[\frac{\pi}{h^2}a^2h^2y + a(b-a)y^2 + \frac{1}{3}(b-a)^2y^3\right]_0^h$$

$$= \frac{\pi}{h^2}a^2h^3 + a(b-a)h^2 + \frac{1}{3}(b-a)^2h^3$$

$$= \pi h\left(a^2 + ab - a^2 + \frac{1}{3}b^2 - \frac{2}{3}ab + \frac{1}{3}a^2\right)$$

$$= \frac{\pi h}{3}(a^2 + ab + b^2)$$

51 **a**



b

Rearranging the curve formulae for the upper right quadrant of both curves:

Parabola: $x^2 = r^2 - y$

Circle: $x^2 = r^2 - y^2$

$$\text{Difference in volumes} = \pi\left(\int_0^r r^2 - y^2\ \mathrm{d}y - \int_0^{r^2} r^2 - y\ \mathrm{d}y\right)$$

$$= \pi\left(\left[r^2y - \frac{1}{3}y^3\right]_0^r - \left[r^2y - \frac{1}{2}y^2\right]_0^{r^2}\right)$$

$$= \pi\left(\frac{2}{3}r^3 - \frac{1}{2}r^4\right)$$

If the difference is zero, then

$$\frac{2}{3}r^3 = \frac{1}{2}r^4$$

$$r = \frac{4}{3}$$

52 **a**

Curve to pass through $(0, 10), (20.5, 25), (50, 17.5), (55, 18)$.

Simultaneous equations:

$$\begin{cases} d = 10 & (1) \\ (20.5)^3 a + (20.5)^2 b + 20.5c + d = 25 & (2) \\ 125000a + 2500b + 50c + d = 17.5 & (3) \\ 166375a + 3025b + 55c + d = 18 & (4) \end{cases}$$

Solving using GDC:

$a = 0.000545, b = -0.0582, c = 1.69, d = 10$

$y = 0.000545x^3 - 0.0582x^2 + 1.69x + 10$

b

$$V = \pi \int_0^{55} y^2 \, dx$$
$$= 74\,400 \text{ cm}^3$$
$$= 74.4 \text{ litres}$$

53 **a**

$k = \tan\theta$ so $1 + k^2 = 1 + \tan^2\theta = \sec^2\theta$

$\sec\theta = \sqrt{1 + k^2}$

b

If $k = \tan\theta$ then $\theta = \arctan k$

$$\text{Shaded area} = \int_0^{\arctan k} x \, dy$$
$$= \int_0^{\theta} \tan y \, dy$$
$$= [\ln \sec y]_0^{\theta}$$
$$= \ln(\sec\theta) - \ln(1)$$
$$= \ln\left(\sqrt{1 + k^2}\right)$$
$$= \frac{1}{2}\ln(1 + k^2)$$

c

The complete rectangle bounded by the axes and the lines $y = \arctan k$ and $x = k$ has area $k \arctan k$

The area between the curve and the x-axis is the remainder after the area from part **b** is removed from this rectangle.

$$\int_0^k \arctan x \, dx = k \arctan k - \frac{1}{2}\ln(1 + k^2)$$

54 **a**

Since the sine of any integer multiple of π has value zero, $x = \frac{1}{n}$ for some $n \in \mathbb{Z}^*$

Then for $0.1 \le x \le 1, x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$

b

Let $u = \frac{\pi}{x}$ so $\mathrm{d}u = -\frac{\pi}{x^2}\mathrm{d}x$

Limits:

When $x = \frac{1}{n+1}, u = (n+1)\pi$

When $x = \frac{1}{n}, u = n\pi$

$$\begin{aligned}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)\,\mathrm{d}x &= \int_{(n+1)\pi}^{n\pi} -\sin u\ \mathrm{d}u \\ &= \int_{n\pi}^{(n+1)\pi} \sin u\ \mathrm{d}u \\ &= [-\cos u]_{n\pi}^{(n+1)\pi} \\ &= \begin{cases} 2 (n \text{ even}) \\ 2\ (n \text{ odd}) \end{cases}\end{aligned}$$

c

$$\begin{aligned}\int_{0.1}^{1} \left|\frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)\right|\,\mathrm{d}x &= \sum_{n=1}^{9}\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \left|\frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)\right|\,\mathrm{d}x\right) (*) \\ &= \sum_{n=1}^{9}\left|\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)\,\mathrm{d}x\right| (**) \\ &= 9(2) \\ &= 18\end{aligned}$$

$(*)$The interval for integration $[0.1, 1]$ can be considered as the union of the intervals $\left[\frac{1}{10}, \frac{1}{9}\right], \left[\frac{1}{9}, \frac{1}{8}\right], \dots, \left[\frac{1}{2}, \frac{1}{1}\right]$, so that the integral over $[0.1, 1]$ is the sum of the integrals over the smaller intervals.

$(**)$ It is legitimate to state that the modulus of the integral is the same as the integral of the modulus in this instance because in each of the intervals the function $\pi x^{-2}\sin(\pi x^{-1})$ is entirely non-negative or entirely non-positive.

55

Implicit differentiation:

$4x + 2y\,y' = 0$

$y' = -\frac{2x}{y}$

The normal at a point (a, b) has gradient $\frac{b}{2a}$ and $b = \pm\sqrt{18 - 2a^2}$

Normal has equation $y - b = \frac{b}{2a}(x - a)$

Substituting $x = 1, y = 0$ to find a condition on a for which the normal passes through $(1, 0)$:

$-b = \frac{b}{2a}(1 - a)$

$\frac{b}{2a}(1 - a + 2a) = 0$

$b = 0$ or $a = -1$

Points are $(\pm 3, 0)$ and $(-1, \pm 4)$

11 Series and differential equations

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Throughout this chapter the worked solutions will adopt the notation y_{GS} for the general solution of y and y_{PS} for a particular solution of y, given initial or boundary conditions.

Exercise 11A

17 **a**

$$\frac{d^2y}{dx^2} = -8e^{2x}$$

$$\frac{dy}{dx} = -4e^{2x} + c$$

$$y_{GS} = -2e^{2x} + cx + d$$

b

$$y(0) = -2 + d = 1 \Rightarrow d = 3$$

$$y'(0) = -4 + c = -2 \Rightarrow c = 2$$

$$y_{PS} = 3 + 2x - 2e^{2x}$$

18

Tip: A table is shown here in the working so that students can check the detail of their calculator output.

A full table of results is not typically needed for this sort of problem unless specifically required in the examination question, but you should lay out the basis by which you generate the values (the iteration formula) and then give the end results, indicating use of GDC.

$$\frac{dy}{dx} = 2x = f(x, y)$$

$$y_{n+1} = y_n + h \times f(x_n, y_n)$$

$$x_0 = y_0 = 0$$

n	x	$y\ (h = 0.1)$	$y\ (h = 0.2)$
0	0	0	0
1	0.1	0	
2	0.2	0.02	0
3	0.3	0.06	
4	0.4	0.12	0.08
5	0.5	0.2	
6	0.6	0.3	0.24
7	0.7	0.42	
8	0.8	0.56	0.48
9	0.9	0.72	
10	1	0.9	0.8

11	1.1	1.1	
12	1.2	1.32	1.2
13	1.3	1.56	
14	1.4	1.82	1.68
15	1.5	2.1	
16	1.6	2.4	2.24
17	1.7	2.72	
18	1.8	3.06	2.88
19	1.9	3.42	
20	2	3.8	3.6

a For $h = 0.1$:

i $y(1) = 0.9$ **ii** $y(2) = 3.8$

b For $h = 0.2$:

i $y(1) = 0.8$ **ii** $y(2) = 3.6$

c

$y_{GS} = x^2 + c$

$y(0) = 0 = c$

$y_{PS} = x^2$

So, in absolute difference, **bii** is the furthest from the true value.

19

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x+y} = f(x, y)$$

$y_{n+1} = y_n + h \times f(x_n, y_n)$

a From GDC: $y(10) \approx 31.4$

b Using a smaller step length would improve the estimate.

20

Calculating distance cumulatively, with

$d_{n+1} = d_n + (t_{n+1} - t_n)v_n$

n	0	1	2	3	4	5
t_n	0	3	6	9	12	15
v_n	0	6	12	19	24	27
d_n	0	0	18	54	111	183

Estimate distance travelled in the first 15 seconds as 183 m

21

Calculating distance cumulatively, with

$d_{n+1} = d_n + (t_{n+1} - t_n)v_n$

n	0	1	2	3	4	5
t_n	0	3	6	9	12	15
v_n	20	15	10	8	6	5
d_n	0	60	105	135	159	177

Estimate distance travelled in the first 15 seconds as 177 m

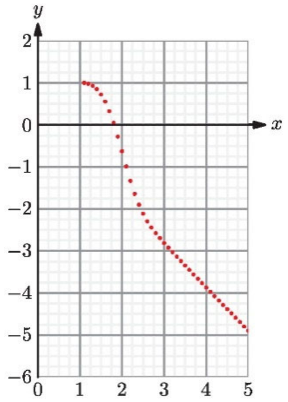
22

$$\frac{dy}{dx} = y^2 - x^2 \ = f(x, y)$$

$y_{n+1} = y_n + h \times f(x_n, y_n)$

$x_0 = 1, y_0 = 1, h = 0.1$

From GDC:



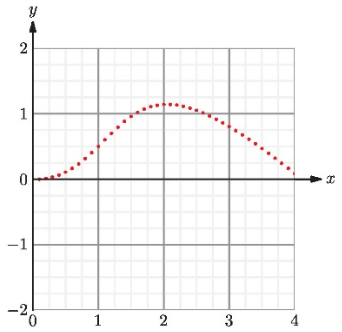
23 **a**

$$\frac{dy}{dx} = \sin(x + y) \ = f(x, y)$$

$y_{n+1} = y_n + h \times f(x_n, y_n)$

$x_0 = 0, y_0 = 0, h = 0.1$

From GDC:



b

From the calculator data table, maximum y for $0 \le x \le 4$ is approximately 1.1

24

Tip: Since the dependent variable in this question is h, we need a new letter for the Euler method step size. Greek letter delta (δ) is often used for small increments.

$$\frac{dh}{dt} = -0.1h^2 - 0.5t = f(h, t)$$

$h_{n+1} = h_n + \delta \times f(t_n, h_n)$

$t_0 = 0, h_0 = 2, \delta = 0.1$

a

From GDC:

$h(1) \approx 1.46$ m

b

In the model using Euler's approximation, $h(2.5) \approx 0.05$ m, $h(2.6) \approx -0.08$ m

It will take approximately 3 seconds for the ash to reach the fire.

25 **a**

$V=\frac{4}{3}\pi r^3, S=4\pi r^2$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}=-kS$

$\frac{dr}{dt}=-k$

When $V=0.5, \frac{dV}{dt}=-0.1$

$\frac{4}{3}\pi r^3=\frac{1}{2}$ so $r=\sqrt[3]{\frac{3}{8\pi}}$

$-k(4\pi r^2)=-0.1$ so $k=\frac{0.1}{4\pi r^2}=0.0328$

b

Integrating:

$r=r(0)-kt$

$r(0)=\sqrt[3]{\frac{3}{8\pi}}$

Then when $r=0, t=\frac{r(0)}{k}=r(0)\div\left(\frac{0.1}{4\pi\left(r(0)\right)^2}\right)=40\pi\left(r(0)\right)^3=40\pi\left(\frac{3}{8\pi}\right)$

$=15$ minutes

26

Let $z=\frac{dy}{dx}$

Then $\frac{dz}{dx}=\frac{d^2y}{dx^2}=-xe^{-x^2}$

Euler's method:

$y(x+h)=y(x)+h\times z(x)$

$z(x+h)=z(x)+h\left(-xe^{-x^2}\right)$

$y(0)=0, z(0)=1, h=0.1$

From GDC, $y(1)\approx 0.904$

27

Let $z=\frac{dy}{dx}$

Then $\frac{dz}{dx}=\frac{d^2y}{dx^2}=2x+y$

Euler's method:

$y(x+h)=y(x)+h\times z(x)$

$z(x+h)=z(x)+h\left(2x+y(x)\right)$

a

$y(0)=1, z(0)=2, h=0.1$

Then

$y(0.1)\approx y(0)+0.1z(0)=1.2$

$\frac{dy}{dx}(0.1)=z(0.1)\approx z(0)+0.1(2\times 0+1)=2.1$

b

From GDC, $y(1)\approx 3.96$

28
Euler's method:
$x(t+h) = x(t) + h \times \dot{x}(t)$
$y(t+h) = y(t) + h \times \dot{y}(t)$
$x(0) = 1, y(0) = 2, h = 0.1$
From GDC, $x(1) \approx 9.46, y(1) \approx 3.71$

Exercise 11B

In many of these solutions, the initial unknown constant will be multiplied by a constant during rearrangement; a new letter is used for the amended constant without further justification, unless the transformation imposes limits upon it. For example, in Q9, the initial constant of integration k is multiplied by 3 to become c. Just as k is an arbitrary constant, so is $3k = c$, so no comment is required.

8
$\int y^{-2}\, dy = \int 1\, dx$
$-y^{-1} = x + c$
$y_{GS} = -\frac{1}{x+c}$

9
$\int y^2\, dy = \int \cos x\, dx$
$\frac{1}{3}y^3 = \sin x + k$
$y_{GS} = \sqrt[3]{3\sin x + c}$

10
$\int e^y\, dy = \int 2x\, dx$
$e^y = x^2 + c$
$y_{GS} = \ln(x^2 + c)$

11 **a**
$\int y^{-1}\, dy = \int \sec^2 x\, dx$
$\ln|y| = \tan x + c$
$|y| = e^c e^{\tan x} = Ae^{\tan x}$, for arbitrary constant $A > 0$
$y = Ae^{\tan x}$, for arbitrary constant A

b
$y(0) = 4 = A$
$y = 4e^{\tan x}$

12
$\int 4y\, dy = \int 9x^2\, dx$
$2y^2 = 3x^3 + c$
$y(0) = 3$
$2(3)^2 = 3(0)^3 + c$
$c = 18$
$2y^2 = 3x^3 + 18$

13

$$\int y^{-1}\ \mathrm{d}y = \int x^{-1}\ \mathrm{d}x$$

$\ln|y| = \ln|x| + c$

$\ln\left|\frac{y}{x}\right| = c$

$\left|\frac{y}{x}\right| = \mathrm{e}^c = K$

$K > 0$ is an arbitrary constant. Removing the modulus restriction,

$\frac{y}{x} = A$

where $A \neq 0$ is an arbitrary constant.

$y_{GS} = Ax, A \neq 0$

Since $y = 0$ is also a solution of the differential equation, the full solution is $y_{GS} = Ax$ with no restriction on constant A.

14

$$\int y^{-2}\ \mathrm{d}y = \int 2x\ \mathrm{d}x$$

$-y^{-1} = x^2 + c$

$y_{GS} = -\frac{1}{x^2 + c}$

15

$$\int y^2\ \mathrm{d}y = \int 3x\ \mathrm{d}x$$

$\frac{1}{3}y^3 = \frac{3}{2}x^2 + c$

$y(2) = 3$

$\frac{1}{3}(3)^3 = \frac{3}{2}(2)^2 + c$

$9 = 6 + c$

$c = 3$

$y^3 = \left(\frac{9}{2}x^2 + 9\right)$

$y_{PS} = \sqrt[3]{\frac{9}{2}(x^2 + 2)}$

16 **a**

$$\int (y-1)^{-1}\ \mathrm{d}y = \int 2x + 4\ \mathrm{d}x$$

$\ln|y - 1| = x^2 + 4x + c$

$|y - 1| = \mathrm{e}^c\mathrm{e}^{x^2+4x} = A\mathrm{e}^{x^2+4x}$, for arbitrary constant $A > 0$

$y - 1 = A\mathrm{e}^{x^2+4x}$, for arbitrary A

$y_{GS} = 1 + A\mathrm{e}^{x^2+4x}$

b

$y(0) = 2 = 1 + A \Rightarrow A = 1$

$y_{PS} = 1 + \mathrm{e}^{x^2+4x}$

17

$$\int \sec^2 y \ \mathrm{d}y = \int \cos x \ \mathrm{d}x$$

$\tan y = \sin x + k$

$\sin x - \tan y = -k = c$

18 **a**

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km$$

When $m = 25, \dfrac{\mathrm{d}m}{\mathrm{d}t} = -5$ so $k = 0.2$

b

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -0.2m$$

$$\int m^{-1} \ \mathrm{d}m = \int -0.2 \ \mathrm{d}t$$

$\ln|m| = -0.2t + c$

$|m| = \mathrm{e}^c \mathrm{e}^{-0.2t} = A\mathrm{e}^{-0.2t}$, for arbitrary constant $A > 0$

$m = A\mathrm{e}^{-0.2t}$ for arbitrary constant A

$m(0) = 25 = A$

$m = 25\mathrm{e}^{-0.2t}$

c

When $m = 12.5, \mathrm{e}^{-0.2t} = 0.5$

$t = -5 \ln 0.5 = 3.47$ seconds

19 **a**

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$

When $N = 2000, \dfrac{d\mathrm{N}}{dt} = 500$

$k = 0.25$

b

$$\int N^{-1} \ \mathrm{d}N = \int 0.25 \ \mathrm{d}t$$

$\ln|N| = 0.25t + c$

$|N| = \mathrm{e}^c \mathrm{e}^{0.25t} = A\mathrm{e}^{0.25t}$, for arbitrary constant $A > 0$

$N = A\mathrm{e}^{0.25t}$

$N(0) = A = 2000$

$N(10) = 2000\mathrm{e}^{2.5} \approx 24\,000$

20 **a**

$$\frac{\mathrm{d}V}{\mathrm{d}t} = kV^{-1}$$

When $V = 300, \dfrac{\mathrm{d}V}{\mathrm{d}t} = 10 \Rightarrow k = 10 \times 300$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3000}{V}$$

b

$$\int V\ \mathrm{d}V = \int 3000\ \mathrm{d}t$$
$$\frac{1}{2}V^2 = 3000t + c$$
$$V_{GS} = \sqrt{6000t + \tilde{c}}$$
$$V(0) = 300 = \sqrt{\tilde{c}} \Rightarrow \tilde{c} = 90000$$
$$V_{PS} = \sqrt{6000t + 90000} = 20\sqrt{15(t + 15)}$$

21 **a**

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v = -0.1(v - 100)$$

Tip: When separating variables, it is often useful to factor out the expression so that the variable taken to the left has a coefficient of 1, and you leave the multiple on the right side. This saves additional rearrangement later.

$$\int (v - 100)^{-1}\,\mathrm{d}v = \int -0.1\ \mathrm{d}t$$
$$\ln|v - 100| = -0.1t + c$$
$|v - 100| = \mathrm{e}^c\mathrm{e}^{-0.1t} = A\mathrm{e}^{-0.1t}$, for arbitrary constant $A > 0$
$$v = 100 + A\mathrm{e}^{-0.1t}$$
$$v(0) = 0 = 100 + A \Rightarrow A = -100$$
$$v = 100(1 - \mathrm{e}^{-0.1t})$$

b)

Velocity is always positive in this model, so distance travelled will be the integral of v over time:

$$d(3) = \int_0^3 v\ \mathrm{d}t = 40.8\ \mathrm{m}$$

22

$$\int y\ \mathrm{d}y = \int 4\mathrm{e}^{-2x}\ \mathrm{d}x$$
$$\frac{1}{2}y^2 = -2\mathrm{e}^{-2x} + c$$
$$y_{GS} = \pm\sqrt{2c - 4\mathrm{e}^{-2x}}$$
$$y(0) = -2 = \pm\sqrt{2c - 4}$$
The curve follows the negative root for the model, $2c - 4 = 4$ so $c = 4$
$$y_{PS} = -2\sqrt{2 - \mathrm{e}^{-2x}}$$

23

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$$
$$\int \mathrm{e}^{-y}\ \mathrm{d}y = \int \mathrm{e}^x\ \mathrm{d}x$$
$$-\mathrm{e}^{-y} = \mathrm{e}^x + c$$
$$-y = \ln(-c - \mathrm{e}^x)$$
$$y_{GS} = -\ln(k - \mathrm{e}^x)$$

24

$$\frac{dy}{dx} = 2e^{x-2y} = 2e^x e^{-2y}$$

$$\int e^{2y}\ dy = \int 2e^x\ dx$$

$$\frac{1}{2}e^{2y} = 2e^x + c$$

$$e^{2y} = 4e^x + k$$

$$y = \frac{1}{2}\ln(k + 4e^x)$$

$$y(0) = 0 = \frac{1}{2}\ln(k+4)\ \text{so}\ k = -3$$

$$y = \frac{1}{2}\ln(4e^x - 3)$$

25

$$\frac{dy}{dx} = xy + 2x + y + 2 = (x+1)(y+2)$$

Separating variables:

$$\int \frac{1}{y+2}\ dy = \int x + 1\ dx$$

$$\ln|y+2| = \frac{1}{2}x^2 + x + c$$

$$y + 2 = Ke^{0.5x^2+x} = K\sqrt{e^{x^2+2x}}$$

$$y_{GS} = K\sqrt{e^{x^2+2x}} - 2$$

$$y(-3) = 0 = K\sqrt{e^3} - 2$$

So $K = 2e^{-1.5}$

$$y_{PS} = 2\sqrt{e^{x^2+2x-3}} - 2$$

$A = 2, B = -2$

26

$$\int y\ dy = \int \sin x\ dx$$

$$\frac{1}{2}y_{GS}^2 = c - \cos x$$

$$y_{GS} = \pm\sqrt{2c - 2\cos x}$$

$y(0) = 10 = \pm\sqrt{2c-2}$ so the curve follows the positive root and $2c = 102$

$$y_{PS} = \sqrt{102 - 2\cos x}$$

27 **a**

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$$\int \sin y\ dy = \int \cos x\ dx$$

$$-\cos y + c = \sin x$$

$$\sin x + \cos y = c$$

b

$y\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$

$\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) = c = 1$

$\sin x + \cos y = 1$

When $x = \frac{\pi}{2}$, $\sin x = 1$

$\cos y = 0$

$y = \pm\frac{\pi}{2}$

28

$y = vx$ so $y' = xv' + v$

$xy' = 2x + 3y$

$x^2v' + xv = 2x + 3xv$

$xv' + v = 2 + 3v$

$xv' = 2 + 2v$

Separating variables:

$\int \frac{1}{1+v}\,\mathrm{d}v = \int \frac{2}{x}\,\mathrm{d}x$

$\ln|1+v| = 2\ln|x| + c = \ln|A|x^2$

$|1+v| = |A|x^2$

Since A is an unrestricted unknown, we can drop the modulus signs.

$1 + \frac{y}{x} = Ax^2$

$y_{GS} = Ax^3 - x$

29 **a**

$xyy' = x^2 + y^2$

Then $y' = \frac{x}{y} + \frac{y}{x} = v + v^{-1}$ where $v = \frac{y}{x}$

Since y' can be expressed as a function solely of (xy^{-1}), this is a homogeneous differential equation.

b

Substitute $y = vx$ so $y' = v + xv'$

$v + xv' = v + v^{-1}$

$xv' = v^{-1}$

Separating variables:

$\int v\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$

$\frac{1}{2}v^2 = \ln|x| + c = \ln|Ax|$

$y^2 = 2x^2\ln|Ax|$

30 **a**

$y' = \frac{(2x+y)^2}{4x^2} = \frac{\left(2+\frac{y}{x}\right)^2}{4} = \frac{(2+v)^2}{4}$ where $v = \frac{y}{x}$

Since y' can be expressed as a function solely of (xy^{-1}), this is a homogeneous differential equation.

b

Substitute $y = vx$ so $y' = v + xv'$

$$v + xv' = \frac{(2+v)^2}{4}$$

$$xv' = \frac{4+v^2}{4}$$

Separating variables:

$$\int \frac{4}{4+v^2}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

$$2\arctan\left(\frac{v}{2}\right) = \ln|x| + c = \ln|Ax|$$

$$\frac{y}{2x} = \tan\left(\frac{1}{2}ln|Ax|\right)$$

$$y_{GS} = 2x\tan\left(\frac{1}{2}ln|Ax|\right)$$

$$y(1) = 0 \Rightarrow 0 = 2\tan\left(\frac{1}{2}ln|A|\right)$$

$\frac{1}{2}\ln|A| = 0$ so set $A = 1$

$$y_{PS} = 2x\tan\left(\frac{1}{2}ln|x|\right)$$

31

$$y' = \frac{x^2+y^2}{x^2+xy} = \frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)}$$

Let $v = \frac{y}{x}$, so $y = vx$ and then $y' = v'x + v$

$$v'x + v = \frac{1+v^2}{1+v}$$

$$v'x = \frac{1-v}{1+v}$$

$$\int \frac{1+v}{1-v}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

$$\int \frac{2}{1-v} - 1\ \mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

$$-2\ln|1-v| - v = \ln|x| + c$$

$$-2\ln\left|1-\frac{y}{x}\right| - \frac{y}{x} = \ln|kx|$$

$$\frac{y}{x} = -\ln\left(\left(1-\frac{y}{x}\right)^2\right) - \ln|Ax| = -\ln\left(\frac{|k|(x-y)^2}{|x|}\right)$$

$$y_{GS} = x\ln\left(\frac{|Ax|}{(x-y)^2}\right)$$

32 **a**

$y' = 2 - \frac{y}{x}$

Let $v = \frac{y}{x}$, so $y = vx$ and then $y' = v'x + v$

$v'x + v = 2 - v$

$v'x = 2 - 2v = -2(v - 1)$

$\int \frac{1}{v-1}\,\mathrm{d}v = \int -\frac{2}{x}\,\mathrm{d}x$

$\ln|v - 1| = -2\ln|x| + k$

$\ln|v - 1| + \ln|Ax^2| = 0$

$\ln|A(xy - x^2)| = 0$

$A(xy - x^2) = \pm 1$

$y = \frac{x^2 \pm A^{-1}}{x}$

Since A is an arbitrary constant of unknown sign, we can generalise:

$y_{GS} = x + \frac{c}{x}$

b

$y(1) = 5 = 1 + c \Rightarrow c = 4$

$y_{PS} = x + \frac{4}{x}$

33

$y' = \frac{y(3x + 4)}{x^2}$

$\int \frac{1}{y}\,\mathrm{d}y = \int 3x^{-1} + 4x^{-2}\,\mathrm{d}x$

$\ln|y| = 3\ln|x| - 4x^{-1} + c$

$\ln\left|\frac{y}{x^3}\right| = c - \frac{4}{x}$

$\frac{y}{x^3} = \mathrm{e}^{c - \frac{4}{x}}$

$y_{GS} = x^3\mathrm{e}^{c - \frac{4}{x}}$

$y(2) = 8 = 8\mathrm{e}^{c-2} \Rightarrow c = 2$

$y_{PS} = x^3\mathrm{e}^{2 - \frac{4}{x}}$

34

$y' = \frac{y^2 + 9}{x}$

$\int \frac{1}{y^2 + 9}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$

$\frac{1}{3}\arctan\left(\frac{y}{3}\right) = \ln|x| + c$

$\arctan\left(\frac{y}{3}\right) = 3\ln|Ax|$

$y = 3\tan(3\ln|Ax|)$

35

$$\frac{dy}{dx} = \frac{2x\sqrt{1-y^2}}{1+x^2}$$

$$\int \frac{1}{\sqrt{1-y^2}}\ \mathrm{d}y = \int \frac{2x}{1+x^2}\ \mathrm{d}x$$

$\arcsin y = \ln|1+x^2| + c$

$y = \sin \ln|1+x^2| + c$

$y(0) = 0, c = 0$

$y = \sin\left(\ln|1+x^2|\right)$

36

$$y' = \frac{y(1+x)}{1-x^2} = \frac{y}{1-x}$$

$(x \neq -1)$

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{1-x}\ \mathrm{d}x$$

$\ln|y| = -\ln|1-x| + c$

$$\left|\frac{y}{1-x}\right| = \mathrm{e}^c$$

$y_{GS} = k(1-x)$

$y(0) = 2 = k$

$y_{PS} = 2(1-x)$

37

$\cos^2 x\, y' = \sec y$

$$\int \cos y\ \mathrm{d}y = \int \sec^2 x\ \mathrm{d}x$$

$\sin y = \tan x + c$

$y_{GS} = \arcsin(\tan x + c)$

38 **a**

$\dot{N} = 0.6N - 0.002N^2 = 0.002(300N - N^2)$

$$\int \frac{1}{300N - N^2}\ \mathrm{d}N = \int 0.002\ \mathrm{d}t$$

$$\int \frac{1}{N(300-N)}\ \mathrm{d}N = 0.002t + c$$

$$\frac{1}{300}\int \left(\frac{1}{N} + \frac{1}{300-N}\right)\ \mathrm{d}N = 0.002t + c$$

$$\ln\left|\frac{N}{300-N}\right| = 0.6t + 300ce$$

$$\frac{N}{300-N} = \mathrm{e}^{0.6t+k}$$

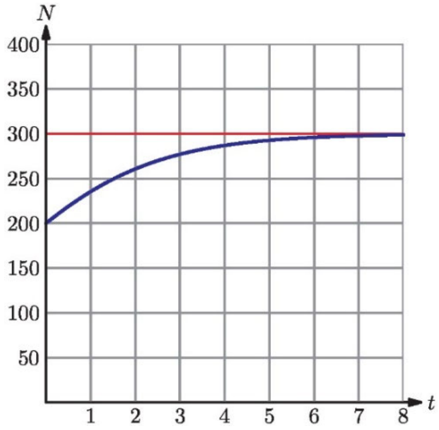
$$N(0) = 200 \Rightarrow \frac{200}{100} = \mathrm{e}^k \Rightarrow \mathrm{e}^k = 2$$

$$\frac{N}{300-N} = 2\mathrm{e}^{0.6t}$$

$N(1 + 2\mathrm{e}^{0.6t}) = 600\mathrm{e}^{0.6t}$

$$N_{PS} = \frac{600\mathrm{e}^{0.6t}}{1+2\mathrm{e}^{0.6t}} = \frac{600}{2+\mathrm{e}^{-0.6t}}$$

b



As $t \to \infty$, N increases towards the limit population 300.

39 **a**

$a = \dot{v} = 10 - 0.1v^2 = -0.1(v^2 - 100)$

$\int \frac{1}{v^2 - 100}\, \mathrm{d}v = \int -0.1\ \mathrm{d}t$

$\frac{1}{20}\int \left(\frac{1}{v - 10} - \frac{1}{v + 10}\right) \mathrm{d}v = \int -0.1\ \mathrm{d}t$

$\ln\left|\frac{v - 10}{v + 10}\right| = -2t + c$

$\frac{v - 10}{v + 10} = A\mathrm{e}^{-2t}$

$v(0) = 0 \Rightarrow \frac{-10}{10} = A \Rightarrow A = -1$

$v(1 + \mathrm{e}^{-2t}) = 10(1 - \mathrm{e}^{2t})$

$v = \frac{10(1 - \mathrm{e}^{-2t})}{1 + \mathrm{e}^{-2t}} = \frac{10(\mathrm{e}^{2t} - 1)}{\mathrm{e}^{2t} + 1}$

b

Let $u = \mathrm{e}^{2t} + 1$ so $\mathrm{d}u = 2\mathrm{e}^{2t}\ \mathrm{d}t = 2(u - 1)\ \mathrm{d}t$

$v = \dot{x} = \frac{10(\mathrm{e}^{2t} - 1)}{\mathrm{e}^{2t} + 1}$

$x = \int \frac{10(\mathrm{e}^{2t} - 1)}{\mathrm{e}^{2t} + 1}\ \mathrm{d}t = \int \frac{10(u - 2)}{u} \times \frac{1}{2(u - 1)}\ \mathrm{d}u$

$x = \int \frac{5(u - 2)}{u(u - 1)}\ \mathrm{d}u$

Partial fractions:

$\frac{5(u - 2)}{u(u - 1)} = \frac{A}{u} + \frac{B}{u - 1}$ for some constants A and B

Multiplying through by the denominator of the LHS:

$5(u - 2) = A(u - 1) + Bu$

$u = 1:\ -5 = B$

$u = 0:\ -10 = -A$

$\frac{5(u - 2)}{u(u - 1)} = \frac{10}{u} - \frac{5}{u - 1}$

$x = \int \frac{10}{u} - \frac{5}{u - 1}\ \mathrm{d}u = 10\ln|u| - 5\ln|u - 1| + c$

At $t = 0$, $x = 0$ and $u = 2$ so $0 = 10\ln 2 + c \Rightarrow c = -10\ln 2$

$$x = 10\ln(e^{2t} + 1) - 5\ln(e^{2t}) - 10\ln 2$$
$$= 5(\ln((e^{2t} + 1)^2) - \ln(e^{2t}) - \ln 4)$$
$$= 5\ln\left(\frac{(e^{2t} + 1)^2}{4e^{2t}}\right)$$

40 **a**

$$y' = xy(y - 1)$$

Let $z = y^{-1}$ so $z' = -y^{-2}y'$

$$z' = -xy^{-1}(y - 1) = -x(1 - y^{-1}) = x(z - 1)$$

b

$$\int \frac{1}{z - 1}\,\mathrm{d}z = \int x\,\mathrm{d}x$$
$$\ln|z - 1| = \frac{1}{2}x^2 + c$$
$$z = 1 + Ae^{\frac{x^2}{2}}$$
$$y_{GS} = \frac{1}{1 + Ae^{\frac{x^2}{2}}}$$
$$y(0) = \frac{1}{3} = \frac{1}{1 + A} \Rightarrow A = 2$$
$$y_{PS} = \frac{1}{1 + 2e^{\frac{x^2}{2}}}$$

41 **a**

$$x^2y' = xy - x + 2$$

Let $u = y + x^{-1}$ so $y = u - x^{-1}$ and $y' = u' + x^{-2}$

$$(u' + x^{-2})x^2 = x(u - x^{-1}) - x + 2$$
$$u'x^2 + 1 = xu - 1 - x + 2$$
$$u'x^2 = xu - x$$
$$xu' = u - 1$$

b

$$\int \frac{1}{u - 1}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$$
$$\ln|u - 1| = \ln|x| + c = \ln(Ax)$$
$$u - 1 = Ax$$
$$y + x^{-1} - 1 = Ax$$
$$y_{GS} = Ax + 1 - x^{-1}$$

42

$(2x - 3y + 3)y' = 2x - 2y + 1$

Let $z = 2x - 3y$ so $z' = 2 - 3y'$

$\frac{(z+3)(2-z')}{3} = 2x - \frac{2(2x-z)}{3} + 1$

$2(z+3) - (z+3)z' = 6x - 4x + 2z + 3$

$-(z+3)z' = 2x - 3$

$\int (z+3)\ \mathrm{d}z = \int 3 - 2x\ \mathrm{d}x$

$\frac{1}{2}(z+3)^2 = 3x - x^2 + c$

$(z+3)^2 = 6x - 2x^2 + c$

When $x = 1, y = 1$ so $z = -1$

$2^2 = 6 - 2 + c \Rightarrow c = 0$

$(z+3)^2 = 6x - 2x^2$

$(2x - 3y + 3)^2 = 6 - 2x^2$

43 **a**

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x - y + 7}{2x + y - 1}$

Let $x = u - 1, y = v + 3$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}$

$4x - y + 7 = 4u - v, 2x + y - 1 = 2u + v$

$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{4u - v}{2u + v} = \frac{4 - \left(\frac{v}{u}\right)}{2 + \left(\frac{v}{u}\right)}$

This is a homogeneous equation because the derivative can be expressed as a function of the ratio of the variables.

b

Let $v = uw$ so $\frac{\mathrm{d}v}{\mathrm{d}u} = u\frac{\mathrm{d}w}{\mathrm{d}u} + w$

$u\frac{\mathrm{d}w}{\mathrm{d}u} + w = \frac{4 - w}{2 + w}$

$u\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{4 - 3w - w^2}{2 + w}$

$\int \frac{2 + w}{(4 + w)(1 - w)}\ \mathrm{d}w = \int \frac{1}{u}\ \mathrm{d}u$

$\frac{2 + w}{(4 + w)(1 - w)} = \frac{A}{4 + w} + \frac{B}{1 - w}$ for some constants A and B

Multiplying through by the denominator of the LHS:

$2 + w = A(1 - w) + B(4 + w)$

$w = 1: 3 = 5B \Rightarrow B = \frac{3}{5}$

$w = -4: \ -2 = 5A \Rightarrow A = -\frac{2}{5}$

$\frac{2 + w}{(4 + w)(1 - w)} = \frac{1}{5}\left(\frac{3}{1 - w} - \frac{2}{4 + w}\right)$

$\int \frac{1}{5}\left(\frac{3}{1 - w} - \frac{2}{4 + w}\right)\ \mathrm{d}w = \int \frac{1}{u}\ \mathrm{d}u$

$-3\ln|1-w| - 2\ln|4+w| + k = 10\ln|u|$
$\ln(|u|^5|1-w|^3(4+w)^2) = k$
$u^5(1-w)^3(4+w)^2 = c$
Substituting back $w = \frac{v}{u}$
$(u-v)^3(4u+v)^2 = c$
Substituting back $u = x+1, v = y-3$:
$(x-y+4)^3(4x+y+1)^2 = c$

44
$y' = \cos^2(x+y) - 1$
Let $z = x+y$ so $z' = 1+y'$
$z' - 1 = \cos^2(z) - 1$
$z' = \cos^2 z$
$$\int \sec^2 z \ \mathrm{d}z = \int 1 \ \mathrm{d}x$$
$\tan z = x + c$
$z = \arctan(x+c)$
$y = z - x = \arctan(x+c) - x$

45
$y'\sqrt{1-x^2} = \sqrt{1-y^2}$
$$\int \frac{1}{\sqrt{1-y^2}} \mathrm{d}y = \int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x$$
$\arcsin y = \arcsin x + c$
Taking sine of both sides and using compound angle formulae:
$$\begin{aligned} y = \sin(arcsin\, x + c) &= x\cos c + \cos(\arcsin x)\sin c \\ &= x\cos c + \sqrt{1-x^2}\sin c \\ &= Cx + D\sqrt{1-x^2} \end{aligned}$$
Where $C = \cos c$ is an unknown constant in $[-1, 1]$ and $D = \pm\sqrt{1-C^2}$ is also a value in $[-1, 1]$
$$y\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2} = \frac{C\sqrt{3}}{2} + \frac{D}{2}$$
$1 - C\sqrt{3} = D$
$1 - 2C\sqrt{3} + 3C^2 = 1 - C^2$
$4C^2 - 2C\sqrt{3} = 0$
$C = 0, D = 1$ or $C = \frac{\sqrt{3}}{2}, D = -\frac{1}{2}$
$A = 0, B = -1$ or $A = \frac{3}{4}, B = -\frac{1}{2}$

46

$\dot{M} = \alpha M^{\frac{2}{3}} - \beta M$

a

At a point of inflection, $\ddot{M} = 0$

$\ddot{M} = \left(\frac{2}{3}\alpha M^{-\frac{1}{3}} - \beta\right)\dot{M}$

Then $\dot{M} = 0$ (which only has solution $M = 0$ or $M = \frac{\alpha}{\beta}$ and is never achieved, given the solution of the differential equation found in part **b**)

Or $\frac{2}{3}\alpha M^{-\frac{1}{3}} - \beta = 0$

$M^{-\frac{1}{3}} = \frac{3\beta}{2\alpha}$

$M = \left(\frac{2\alpha}{3\beta}\right)^3 = \frac{8\alpha^3}{27\beta^3}$

b

Let $v = M^{\frac{1}{3}}$ so $\dot{v} = \frac{1}{3}M^{-\frac{2}{3}}\dot{M}$

$\dot{v} = \frac{1}{3}(\alpha - \beta v) = -\frac{\beta}{3}\left(v - \frac{\alpha}{\beta}\right)$

$\int \frac{1}{v - \frac{\alpha}{\beta}}\,\mathrm{d}v = \int -\frac{\beta}{3}\,\mathrm{d}t$

$\ln\left|v - \frac{\alpha}{\beta}\right| = -\frac{\beta t}{3} + c$

$v - \frac{\alpha}{\beta} = k\mathrm{e}^{-\frac{\beta t}{3}}$

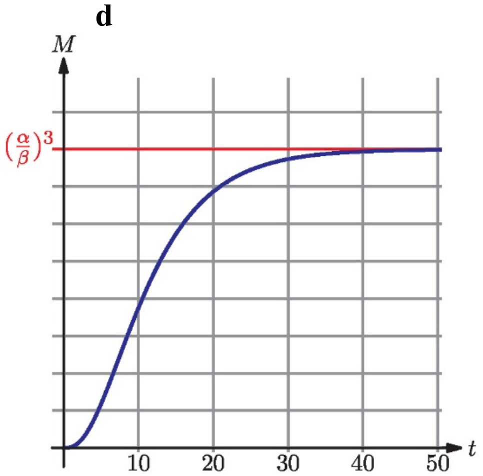
$v = \frac{\alpha}{\beta} + k\mathrm{e}^{-\frac{\beta t}{3}}$

$M = \left(\frac{\alpha}{\beta} + k\mathrm{e}^{-\frac{\beta t}{3}}\right)^3$

c

As $t \to \infty, M \to \left(\frac{\alpha}{\beta}\right)^3$

d



Exercise 11C

5 **a** $\mu = e^{\int 3\,dx} = e^{3x}$

b

Multiplying both sides by the integrating factor:

$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{4x}$

$\frac{d}{dx}(e^{3x}y) = e^{4x}$

$e^{3x}y = \frac{1}{4}e^{4x} + c$

$y_{GS} = \frac{1}{4}e^{x} + ce^{-3x}$

6

$\mu = e^{\int -2\,dx} = e^{-2x}$

Multiplying both sides by the integrating factor:

$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = e^{-x}$

$\frac{d}{dx}(e^{-2x}y) = e^{-x}$

$e^{-2x}y = -e^{-x} + c$

$y_{GS} = -e^{x} + ce^{2x}$

7 **a** $\mu = e^{\int 2x\,dx} = e^{x^2}$

b

Multiplying both sides by the integrating factor:

$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 1 + 8xe^{x^2}$

$\frac{d}{dx}\left(2xe^{x^2}y\right) = 1 + 8xe^{x^2}$

$e^{x^2}y = x + 4e^{x^2} + c$

$y_{GS} = xe^{-x^2} + 4 + ce^{-x^2}$

8 **a** $\mu = e^{\int 4x\,dx} = e^{2x^2}$

b

Multiplying both sides by the integrating factor:

$e^{2x^2}\frac{dy}{dx} + 4xe^{2x^2}y = 5 + 12xe^{2x^2}$

$\frac{d}{dx}\left(e^{2x^2}y\right) = 5 + 12xe^{2x^2}$

$e^{2x^2}y = 5x + 3e^{2x^2} + c$

$y_{GS} = 5xe^{-2x^2} + 3 + ce^{-2x^2}$

9 **a** $\mu = e^{\int -3x^{-1}\,dx} = e^{-3\ln x} = \left(e^{\ln x}\right)^{-3} = x^{-3}$

b

Multiplying both sides by the integrating factor:

$x^{-3}\frac{dy}{dx} - 3x^{-4}y = 2x^{-1}$

$\frac{d}{dx}(x^{-3}y) = 2x^{-1}$

$x^{-3}y = 2\ln|x| + c$

$y_{GS} = 2x^3\ln|x| + cx^3$

10 **a** $\mu = \mathrm{e}^{\int 2x^{-1}\,\mathrm{d}x} = \mathrm{e}^{2\ln x} = \left(\mathrm{e}^{\ln x}\right)^2 = x^2$

b

Multiplying both sides by the integrating factor:

$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 12x^3$

$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = 12x^3$

$x^2y = 3x^4 + c$

$y_{GS} = 3x^2 + cx^{-2}$

$y(1) = 5 = 3 + c \Rightarrow c = 2$

$y_{PS} = 3x^2 + 2x^{-2}$

11

$\mu = \mathrm{e}^{\int \sin x\,\mathrm{d}x} = \mathrm{e}^{-\cos x}$

Multiplying both sides by the integrating factor:

$\mathrm{e}^{-\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} + \sin x\,\mathrm{e}^{-\cos x}y = 2$

$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-\cos x}y) = 2$

$\mathrm{e}^{-\cos x}y = 2x + c$

$y_{GS} = (2x + c)\mathrm{e}^{\cos x}$

12

$\mu = \mathrm{e}^{\int 10x\,\mathrm{d}x} = \mathrm{e}^{5x^2}$

Multiplying both sides by the integrating factor:

$\mathrm{e}^{5x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 10x\mathrm{e}^{5x^2}y = 3$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{5x^2}y\right) = 3$

$\mathrm{e}^{5x^2}y = 3x + c$

$y_{GS} = (3x + c)\mathrm{e}^{-5x^2}$

$y(0) = 4 = c$

$y_{PS} = (3x + 4)\mathrm{e}^{-5x^2}$

13

$\mu = \mathrm{e}^{\int x^{-1}\,\mathrm{d}x} = \mathrm{e}^{\ln x} = x$

b)

Multiplying both sides by the integrating factor:

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^{-2}$

$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^{-2}$

$xy = -x^{-1} + c$

$y_{GS} = -x^{-2} + cx^{-1}$

14 **a** $\mu = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

b

Multiplying both sides by the integrating factor:

$$\sec x \frac{dy}{dx} + \sec x \tan x \, y = \sec^2 x$$

$$\frac{d}{dx}(\sec x \, y) = \sec^2 x$$

$$\sec x \, y = \tan x + c$$

$$y_{GS} = \sin x + c \cos x$$

15 **a** $\mu = e^{\int -2\tan x \, dx} = e^{-2\ln \sec x} = \left(e^{\ln \sec x}\right)^{-2} = (\sec x)^{-2} = \cos^2 x$

b

Multiplying both sides by the integrating factor:

$$\cos^2 x \frac{dy}{dx} - 2\sin x \cos x \, y = 1$$

$$\frac{d}{dx}(\cos^2 x \, y) = 1$$

$$\cos^2 x \, y = x + c$$

$$y_{GS} = (x + c)\sec^2 x$$

$$y\left(\frac{\pi}{4}\right) = 8 = \left(\frac{\pi}{4} + c\right) \times 2$$

$$c = 4 - \frac{\pi}{4}$$

$$y_{PS} = \left(x + 4 - \frac{\pi}{4}\right)\sec^2 x$$

16 **a** $\mu = e^{\int \cos x \, dx} = e^{\sin x}$

b

Multiplying both sides by the integrating factor:

$$e^{\sin x} \frac{dy}{dx} + e^{\sin x} \cos x \, y = e^{\sin x} \cos x$$

$$\frac{d}{dx}\left(e^{\sin x} y\right) = e^{\sin x} \cos x$$

$$e^{\sin x} y = e^{\sin x} + c$$

$$y_{GS} = 1 + ce^{-\sin x}$$

17 **a** $\mu = e^{\int x^{-1} \, dx} = e^{\ln x} = x$

b

Multiplying both sides by the integrating factor:

$$x \frac{dy}{dx} + y = x^{-1}$$

$$\frac{d}{dx}(xy) = x^{-1}$$

$$xy = \ln|x| + c$$

$$y_{GS} = \frac{\ln|x| + c}{x}$$

$$y(1) = 2 = c$$

$$y_{PS} = \frac{\ln|x| + 2}{x}$$

18

$\mu = e^{\int (x+2)^{-1}\,dx} = e^{\ln(x+2)} = x + 2$

Multiplying both sides by the integrating factor:

$(x+2)\frac{dy}{dx} + y = x^2 + x - 2$

$\frac{d}{dx}\big((x+2)y\big) = x^2 + x - 2$

$(x+2)y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + k = \frac{2x^3 + 3x^2 - 12x + c}{6}$

$y_{GS} = \frac{2x^3 + 3x^2 - 12x + c}{6(x+2)}$

19

$\mu = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\ln(x^2+1)} = x^2 + 1$

Multiplying both sides by the integrating factor:

$(x^2+1)\frac{dy}{dx} + 2xy = 2x^3 + 2x$

$\frac{d}{dx}\big((x^2+1)y\big) = 2x^3 + 2x$

$(x^2+1)y = \frac{1}{2}x^4 + x^2 + k = \frac{x^4 + 2x^2 + c}{2}$

$y_{GS} = \frac{x^4 + 2x^2 + c}{2(x^2+1)}$

$y(0) = 0 = \frac{c}{2}$

$y_{PS} = \frac{x^4 + 2x^2}{2(x^2+1)}$

20

$\frac{dy}{dx} + 2x^{-1}y = 4x^{-1}$

$\mu = e^{\int 2x^{-1}\,dx} = e^{2\ln x} = \left(e^{\ln x}\right)^2 = x^2$

Multiplying both sides by the integrating factor:

$x^2\frac{dy}{dx} + 2xy = 4x$

$\frac{d}{dx}(x^2y) = 4x$

$x^2y = 2x^2 + c$

$y_{GS} = 2 + cx^{-2}$

21

Tip: If you do not recognise that the equation already has a perfect derivative on the left side, rearrange and find the integrating factor in the normal way. If you do observe this fact, you can integrate immediately but be clear about your reasoning.

$\frac{dy}{dx} + \frac{y}{(x-1)} = \frac{6x}{(x-1)}$

$\mu = e^{\int (x-1)^{-1}\,dx} = e^{\ln(x-1)} = x - 1$

Multiplying both sides by the integrating factor:

$$(x-1)\frac{dy}{dx} + y = 6x$$

$$\frac{d}{dx}\big((x-1)y\big) = 6x$$

$$(x-1)y = 3x^2 + c$$

$$y_{GS} = \frac{3x^2 + c}{x-1}$$

22

$$\mu = e^{\int -\tan x \, dx} = e^{\ln(\cos x)} = \cos x$$

Multiplying both sides by the integrating factor:

$$\cos x \frac{dy}{dx} - \sin x \, y = \cos^2 x$$

$$\frac{d}{dx}(\cos x \, y) = \cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

$$\cos x \, y = \frac{1}{4}\sin 2x + \frac{1}{2}x + c = \frac{1}{2}\sin x \cos x + \frac{1}{2}x + c$$

$$y_{GS} = \frac{1}{2}\sin x + \left(\frac{1}{2}x + c\right)\sec x$$

23

Left side is a perfect derivative

$$\frac{d}{dx}(x^2 y) = e^x$$

$$x^2 y = e^x + c$$

$$y_{GS} = \frac{e^x + c}{x^2}$$

$$y(1) = 0 = e + c \Rightarrow c = -e$$

$$y_{PS} = \frac{e^x - e}{x^2}$$

Then $y(2) = \frac{e^2 - e}{4}$

24

$$\frac{dy}{dx} - 2\tan x \, y = 3\sec x$$

$$\mu = e^{\int -2\tan x \, dx} = e^{2\ln(\cos x)} = \cos^2 x$$

Multiplying both sides by the integrating factor:

$$\cos^2 x \frac{dy}{dx} - 2\sin x \cos x \, y = 3\cos x$$

$$\frac{d}{dx}(\cos^2 x \, y) = 3\cos x$$

$$\cos^2 x \, y = 3\sin x + c$$

$$y_{GS} = (3\sin x + c)\sec^2 x$$

25

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \cot x\, y = 1$$

$\mu = \mathrm{e}^{\int \cot x \, \mathrm{d}x} = \mathrm{e}^{\ln \sin x} = \sin x$

Multiplying both sides by the integrating factor:

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + \cos x\, y = \sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x\, y) = \sin x$$

$\sin x\, y = -\cos x + c$

$y_{GS} = -\cot x + c \operatorname{cosec} x$

$y\left(\frac{\pi}{4}\right) = 1 = -1 + c\sqrt{2} \Rightarrow c = \sqrt{2}$

$y_{PS} = \sqrt{2} \operatorname{cosec} x - \cot x$

26

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x} y = \frac{x^2}{x-3}$$

$\mu = \mathrm{e}^{\int -2x^{-1} \, \mathrm{d}x} = \mathrm{e}^{-2\ln x} = x^{-2}$

Multiplying both sides by the integrating factor:

$$x^{-2} \frac{\mathrm{d}y}{\mathrm{d}x} - 2x^{-3} y = \frac{1}{x-3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2} y) = \frac{1}{x-3}$$

$x^{-2} y = \ln|x-3| + c$

$y_{GS} = x^2(\ln|x-3| + c)$

27

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \tan x\, y = \cos x$$

$\mu = \mathrm{e}^{\int \tan x \, \mathrm{d}x} = \mathrm{e}^{\ln \sec x} = \sec x$

Multiplying both sides by the integrating factor:

$$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} + \sec x \tan x\, y = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x\, y) = 1$$

$\sec x\, y = x + c$

$y_{GS} = \cos x\,(x + c)$

$y(0) = 2 = c$

$y_{PS} = (x + 2) \cos x$

28

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{(x-x^{-1})}y=\frac{1}{(x-x^{-1})}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2x}{(x^2-1)}y=\frac{x}{(x^2-1)}$$
$$\mu=\mathrm{e}^{\int\left(\frac{2x}{x^2-1}\right)\mathrm{d}x}=\mathrm{e}^{\ln(x^2-1)}=x^2-1$$
Multiplying both sides by the integrating factor:
$$(x^2-1)\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-1)y\right)=x$$
$$(x^2-1)y=\frac{1}{2}x^2+k=\frac{x^2+c}{2}$$
$$y_{GS}=\frac{x^2+c}{2(x^2-1)}$$

29 **a** $\frac{\mathrm{d}v}{\mathrm{d}t}+2v=10$

ai

$$\mu=\mathrm{e}^{\int 2\,\mathrm{d}t}=\mathrm{e}^{2t}$$
Multiplying both sides by the integrating factor:
$$\mathrm{e}^{2t}\frac{\mathrm{d}v}{\mathrm{d}t}+2e^{2t}v=10\mathrm{e}^{2t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{2t}v)=10\mathrm{e}^{2t}$$
$$\mathrm{e}^{2t}v=5\mathrm{e}^{2t}+c$$
$$v_{GS}=5+c\mathrm{e}^{-2t}$$
$$v(0)=v_0=5+c\Rightarrow c=v_0-5$$
$$v_{PS}=5+(v_0-5)\mathrm{e}^{-2t}$$

aii $v\to 5$ as $t\to\infty$

b $\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{1}{t+1}v=10$

bi

$$\mu=\mathrm{e}^{\int(t+1)^{-1}\,\mathrm{d}t}=\mathrm{e}^{\ln(t+1)}=t+1$$
Multiplying both sides by the integrating factor:
$$(t+1)\frac{\mathrm{d}v}{\mathrm{d}t}+v=10(t+1)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left((t+1)v\right)=10(t+1)$$
$$(t+1)v=5(\mathrm{t}+1)^2+c$$
$$v_{GS}=5(t+1)+\frac{c}{t+1}$$
$$v(0)=v_0=5+c\Rightarrow c=v_0-5$$
$$v_{PS}=5(t+1)+\frac{(v_0-5)}{t+1}$$

bii Then

$$\frac{\mathrm{d}v}{\mathrm{d}t}=5-\frac{v_0-5}{(t+1)^2}$$
So $\frac{\mathrm{d}v}{\mathrm{d}t}\to 5$ as $t\to\infty$

Long term acceleration is $5\ \mathrm{m\ s^{-2}}$

30 **a**

Using product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y^2) = 2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{1}{xy}$$

Multiplying both sides by $2x^2y$:

$$2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y^2) = 2x$$

$$x^2y^2 = x^2 + c$$

$$y^2 = 1 + cx^{-2}$$

$$y = \sqrt{1 + cx^{-2}} \text{ or } -\sqrt{1 + cx^{-2}}$$

31 **a**

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3 = \mathrm{e}^x$$

Let $z = y^3$

Then $\frac{\mathrm{d}z}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$ and $z = y^3$

$$x\frac{\mathrm{d}z}{\mathrm{d}x} + z = \mathrm{e}^x$$

b

Left side is a perfect derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(xz) = \mathrm{e}^x$$

$$xz = \mathrm{e}^x + c$$

$$z_{GS} = \frac{\mathrm{e}^x + c}{x}$$

$$y = \sqrt[3]{z} \text{ so } y_{Gs} = \sqrt[3]{\frac{\mathrm{e}^x + c}{x}}$$

32 **a**

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y^2}{x} = x^2$$

Let $z = y^2$ so $\frac{\mathrm{d}z}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$

Substituting:

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}z = x^2$$

b

$\mu = \mathrm{e}^{-\int x^{-1}\,\mathrm{d}x} = \mathrm{e}^{-\ln x} = x^{-1}$

Multiplying both sides by the integrating factor:

$x^{-1}\frac{\mathrm{d}z}{\mathrm{d}x} - x^{-2}z = x$

$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}z) = x$

$x^{-1}z = \frac{1}{2}x^2 + c$

$z_{GS} = \frac{1}{2}x^3 + cx$

$y = \sqrt{z}$

$y_{GS} = \sqrt{\frac{1}{2}x^3 + cx}$ or $-\sqrt{\frac{1}{2}x^3 + cx}$

$y(2) = -2 = -\sqrt{4 + 2c}$ (using the negative root) $\Rightarrow c = 0$

$y_{PS} = -\sqrt{\frac{x^3}{2}}$

33

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{2xy}$

Let $y = \sqrt{u}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x}$ and $y^2 = u$

$\frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{(x^2 + u)}{2x\sqrt{u}}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{(x^2 + u)}{x} = x + x^{-1}u$

$\frac{\mathrm{d}u}{\mathrm{d}x} - x^{-1}u = x$

$\mu = \mathrm{e}^{-\int x^{-1}\,\mathrm{d}x} = \mathrm{e}^{-\ln x} = x^{-1}$

Multiplying both sides by the integrating factor:

$x^{-1}\frac{\mathrm{d}u}{\mathrm{d}x} - x^{-2}u = 1$

$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}u) = 1$

$x^{-1}u = x + c$

$u_{GS} = x^2 + cx$

$y_{GS} = \sqrt{x^2 + cx}$

34 **a**

$\cos y\frac{\mathrm{d}y}{\mathrm{d}x} + \tan x \sin y = 2\cos^2 x$

Let $z = \sin y$ so $\frac{\mathrm{d}z}{\mathrm{d}x} = \cos y\frac{\mathrm{d}y}{\mathrm{d}x}$

Substituting:

$\frac{\mathrm{d}z}{\mathrm{d}x} + \tan x\, z = 2\cos^2 x$

b

$\mu = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x$

Multiplying both sides by the integrating factor:

$\sec x \frac{dz}{dx} + \sec x \tan x\, z = 2\cos x$

$\frac{d}{dx}(\sec x\, z) = 2\cos x$

$\sec x\, z = 2\sin x + c$

$z_{GS} = 2\sin x \cos x + c\cos x = \sin 2x + c\cos x$

$z = \sin y$

$\sin 2x - \sin y = -c\cos x$

$y\left(\frac{\pi}{4}\right) = \frac{\pi}{6}$

$1 - \frac{1}{2} = -\frac{c}{\sqrt{2}}$ so $c = -\frac{1}{\sqrt{2}}$

Then $\sin 2x - \sin y = \frac{\sqrt{2}}{2}\cos x$

35 **a**

$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = x$

Let $z = x\frac{dy}{dx}$ so $\frac{dz}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2}$

Then $\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{dz}{dx} - \frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{dz}{dx} - \frac{1}{x}z\right)$

Substituting:

$\frac{1}{x}\left(\frac{dz}{dx} - \frac{1}{x}z\right) - \frac{1}{x^2}z = x$

$\frac{dz}{dx} - \frac{2}{x}z = x^2$

b

$\mu = e^{\int -2x^{-1}\, dx} = e^{-2\ln x} = x^{-2}$

Multiplying both sides by the integrating factor:

$x^{-2}\frac{dz}{dx} - 2x^{-3}z = 1$

$\frac{d}{dx}(x^{-2}z) = 1$

$x^{-2}z = x + c$

$z_{GS} = x^3 + cx^2$

c

$x\frac{dy}{dx} = x^3 + cx^2$

$\frac{dy}{dx} = x^2 + cx$

$y_{GS} = \frac{1}{3}x^3 + \frac{1}{2}cx^2 + d$

Since half an unknown constant is still an unknown constant, relabelling:

$y_{GS} = \frac{1}{3}x^3 + \tilde{c}x^2 + d$

Worked solutions

36 $\frac{dP}{dt} + \beta P = \alpha e^{-\gamma t}$

ai

$\mu = e^{\int \beta \, dt} = e^{\beta t}$

Multiplying both sides by the integrating factor:

$e^{\beta t}\frac{dP}{dt} + \beta e^{\beta t}P = \alpha e^{(\beta-\gamma)t}$

$\frac{d}{dt}\left(e^{\beta t}P\right) = \alpha e^{(\beta-\gamma)t}$

$e^{\beta t}P = \frac{\alpha}{\beta-\gamma}e^{(\beta-\gamma)t} + c$

$P_{GS} = \frac{\alpha}{\beta-\gamma}e^{-\gamma t} + ce^{-\beta t}$

$P(0) = 0 = \frac{\alpha}{\beta-\gamma} + c \Rightarrow c = -\frac{\alpha}{\beta-\gamma}$

$P_{PS} = \frac{\alpha}{\beta-\gamma}\left(e^{-\gamma t} - e^{-\beta t}\right)$

aii

Stationary point where

$\frac{dP}{dt} = \alpha e^{-\gamma t} - \beta P = 0$

$\alpha e^{-\gamma t} = \frac{\alpha\beta}{\beta-\gamma}\left(e^{-\gamma t} - e^{-\beta t}\right)$

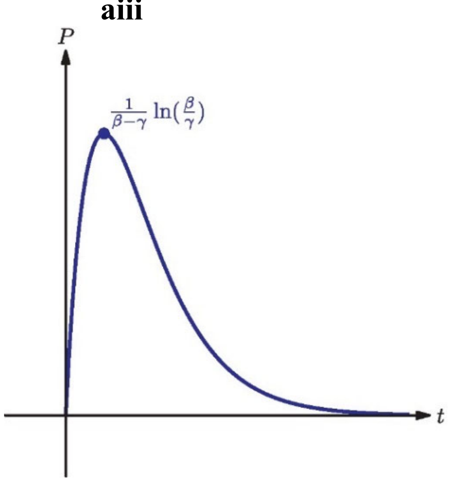
$(\beta-\gamma)e^{-\gamma t} = \beta e^{-\gamma t} - \beta e^{-\beta t}$

$\gamma e^{-\gamma t} = \beta e^{-\beta t}$

$e^{(\beta-\gamma)t} = \frac{\beta}{\gamma}$

$t = \frac{1}{\beta-\gamma}\ln\left(\frac{\beta}{\gamma}\right)$

aiii



b $\frac{dP}{dt} + \beta P = \alpha e^{-\beta t}$

$\mu = e^{\int \beta \, dt} = e^{\beta t}$

Multiplying both sides by the integrating factor:

$$e^{\beta t}\frac{dP}{dt} + \beta e^{\beta t}P = \alpha$$

$$\frac{d}{dt}\left(e^{\beta t}P\right) = \alpha$$

$$e^{\beta t}P = \alpha t + c$$

$$P_{GS} = (\alpha t + c)e^{-\beta t}$$

$$P(0) = 0 = c \Rightarrow c = 0$$

$$P_{PS} = \alpha t e^{-\beta t}$$

37 $\frac{d[\mathrm{Bi}]}{dt} = -k_1[\mathrm{Bi}]$

a

Separating variables:

$$\int \frac{1}{[\mathrm{Bi}]} \, d[\mathrm{Bi}] = \int -k_1 \, dt$$

$$\ln[\mathrm{Bi}] = -k_1 t + c$$

$$[\mathrm{Bi}] = Ae^{-k_1 t}$$

$$[\mathrm{Bi}](0) = [\mathrm{Bi}]_0 = A$$

$$[\mathrm{Bi}] = [\mathrm{Bi}]_0 e^{-k_1 t}$$

b

The rate of change in amount of Polonium is the resultant of the rate of gain due to decay of Bismuth and the rate of loss due to decay of Polonium.

c

$$\frac{d[\mathrm{Po}]}{dt} = k_1[\mathrm{Bi}] - k_2[\mathrm{Po}] = k_1[\mathrm{Bi}]_0 e^{-k_1 t} - k_2[\mathrm{Po}]$$

$$\frac{d[\mathrm{Po}]}{dt} + k_2[\mathrm{Po}] = k_1[\mathrm{Bi}]_0 e^{-k_1 t}$$

This is a linear differential equation for [Po].

$\mu = e^{\int k_2 \, dt} = e^{k_2 t}$

Multiplying both sides by the integrating factor:

$$e^{k_2 t}\frac{d[\mathrm{Po}]}{dt} + k_2 e^{k_2 t}[\mathrm{Po}] = k_1[\mathrm{Bi}]_0 e^{(k_2 - k_1)t}$$

$$\frac{d}{dt}\left(e^{k_2 t}[\mathrm{Po}]\right) = k_1[\mathrm{Bi}]_0 e^{(k_2 - k_1)t}$$

$$e^{k_2 t}[\mathrm{Po}] = \frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0 e^{(k_2 - k_1)t} + c$$

$$[\mathrm{Po}]_{GS} = \frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0 e^{-k_1 t} + ce^{-k_2 t}$$

$$[\mathrm{Po}](0) = 0 = c + \frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0 \Rightarrow c = -\frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0$$

$$[\mathrm{Po}]_{PS} = \frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0\left(e^{-k_1 t} - e^{-k_2 t}\right)$$

d

Since the total concentration [Bi] + [Po] + [Pb] must always equal the initial concentration $[\mathrm{Bi}]_0$,

$$[\mathrm{Pb}] = [\mathrm{Bi}]_0 - [\mathrm{Bi}] - [\mathrm{Po}]$$

$$= [\mathrm{Bi}]_0 - [\mathrm{Bi}]_0 \mathrm{e}^{-k_1 t} - \frac{k_1}{k_2 - k_1}[\mathrm{Bi}]_0(\mathrm{e}^{-k_1 t} - \mathrm{e}^{-k_2 t})$$

$$= [\mathrm{Bi}]_0 \left(1 - \frac{k_2}{k_2 - k_1}\mathrm{e}^{-k_1 t} - \frac{k_1}{k_2 - k_1}\mathrm{e}^{-k_2 t}\right)$$

e

As $t \to \infty$, $[\mathrm{Pb}] \to [\mathrm{Bi}]_0$

This is to say, in the long term, all the Bismuth will decay down to lead.

Exercise 11D

Tip at start of exercise: Throughout these worked solutions you will see the ‘order’ notation $O(n)$. This means that terms in x^n and greater powers are being discarded in the approximation, and you may also see this notation in some past paper mark schemes.

Although this notation is not required, it is useful to be precise about exactly which terms you are discarding, particularly when combining expansions using addition, multiplication or division, so that you can be confident of the level of precision in your final answer, and so that you do not leave out any necessary terms or parts of coefficients. It also allows the use of the equals sign = instead of approximately equals ≈.

12 **a**

$\mathrm{f}'(x) = \sec^2 x$

$\mathrm{f}''(x) = 2\sec^2 x \tan x$

$\mathrm{f}'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

b

$\mathrm{f}(0) = 0$

$\mathrm{f}'(0) = 1$

$\mathrm{f}''(0) = 0$

$\mathrm{f}'''(0) = 2$

Maclaurin series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + O(x^4)$$

$$\mathrm{f}(x) = x + \frac{1}{3}x^3 + O(x^4)$$

c

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$

$$= 0.849\%$$

13 **a**

$\mathrm{f}(x) = \sec x$

$\mathrm{f}'(x) = \sec x \tan x$

$\mathrm{f}''(x) = \sec x \tan^2 x + \sec^3 x$

$\mathrm{f}'''(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$

b

$f(0) = 1$
$f'(0) = 0$
$f''(0) = 1$
$f'''(0) = 0$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$f(x) = 1 + \frac{1}{2}x^2 + O(x^4)$$

c

$$\sec(0.2) \approx 1 + \frac{1}{2} \times 0.04 = 1.02$$

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$
$$= 0.0332\%$$

14

$f(x) = x\cos 3x$
$f'(x) = -3x\sin 3x + \cos 3x$
$f''(x) = -9x\cos 3x - 6\sin 3x$
$f'''(x) = 27x\sin 3x - 27\cos 3x$
$f^{(4)}(x) = 81x\cos 3x + 108\sin 3x$
$f^{(5)}(x) = -243x\sin 3x + 405\cos 3x$
$f(0) = 0$
$f'(0) = 1$
$f''(0) = 0$
$f'''(0) = -27$
$f^{(4)}(0) = 0$
$f^{(5)}(0) = 405$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + O(x^6)$$

$$f(x) = x - \frac{27}{6}x^3 + \frac{405}{120}x^5 + O(x^6)$$

$$f(x) = x - \frac{9}{2}x^3 + \frac{27}{8}x^5 + O(x^6)$$

15 **a**

$$\ln(1-x) = (-x) + \frac{1}{2}(-x)^2 + \frac{1}{3}(-x)^3 + O(x^4)$$
$$= -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$$

b

When $x = \frac{1}{10}$:

$$\ln(0.9) \approx -\frac{1}{10} + \frac{1}{200} - \frac{1}{3000}$$
$$\approx \frac{-300 + 15 - 1}{3000} = -\frac{143}{1500}$$

16 **a**

$f(x) = \ln(e + x)$
$f'(x) = (e + x)^{-1}$
$f''(x) = -(e + x)^{-2}$
$f'''(x) = 2(e + x)^{-3}$
$f(0) = 1$
$f'(0) = e^{-1}$
$f''(0) = -e^{-2}$
$f'''(0) = 2e^{-3}$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$f(x) = 1 + \frac{1}{e}x - \frac{1}{2e^2}x^2 + \frac{1}{3e^3}x^3 + O(x^4)$$

b

$$\ln(1 + e) \approx 1 + \frac{1}{e} - \frac{1}{2e^2} + \frac{1}{3e^3}$$
$$\approx \frac{6e^3 + 6e^2 - 3e + 2}{6e^3}$$

c

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$
$$= 0.27\%$$

17

$f(x) = \ln(1 + x + x^2)$

$$f'(x) = \frac{1 + 2x}{1 + x + x^2}$$

$$f''(x) = \frac{2(1 + x + x^2) - (1 + 2x)^2}{(1 + x + x^2)^2} = \frac{1 - 2x - 2x^2}{(1 + x + x^2)^2} = \frac{3}{(1 + x + x^2)^2} - \frac{2}{1 + x + x^2}$$

$$f'''(x) = -\frac{6(1 + 2x)}{(1 + x + x^2)^3} + \frac{2(1 + 2x)}{(1 + x + x^2)^2}$$

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 1$
$f'''(0) = -4$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$f(x) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + O(x^4)$$

18

$f(x) = (1 + x)\sin 2x$
$f'(x) = \sin 2x + 2(1 + x)\cos 2x$
$f''(x) = 4\cos 2x - 4(1 + x)\sin 2x$
$f'''(x) = -12\sin 2x - 8(1 + x)\cos 2x$
$f^{(4)}(x) = -32\cos 2x + 16(1 + x)\sin 2x$
$f(0) = 0$
$f'(0) = 2$
$f''(0) = 4$
$f'''(0) = -8$
$f^{(4)}(0) = -32$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + O(x^5)$$

$$f(x) = 2x + 2x^2 - \frac{4}{3}x^3 - \frac{4}{3}x^4 + O(x^5)$$

19 Tip: For a question of this sort you can either use the standard known Maclaurin series for e^x and multiply by the polynomial, or you can use differentiation and calculate coefficients that way. The choice is really a matter of convenience unless the question specifies. The second method is shown here.

a

$f(x) = (1 + x + x^2)e^x$
$f'(x) = (2 + 3x + x^2)e^x$
$f''(x) = (5 + 5x + x^2)e^x$
$f'''(x) = (10 + 7x + x^2)e^x$
$f(0) = 1$
$f'(0) = 2$
$f''(0) = 5$
$f'''(0) = 10$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$f(x) = 1 + 2x + \frac{5}{2}x^2 + \frac{5}{3}x^3 + O(x^4)$$

b

$f(x) = e^{x+x^2}$
$f'(x) = (1 + 2x)e^{x+x^2}$
$f''(x) = \big(2 + (1 + 2x)(1 + 2x)\big)e^{x+x^2} = (3 + 4x + 4x^2)e^{x+x^2}$
$f'''(x) = \big(4 + 8x + (3 + 4x + 4x^2)(1 + 2x)\big)e^{x+x^2} = \big(7 + O(x)\big)e^{x+x^2}$
$f(0) = 1$
$f'(0) = 1$
$f''(0) = 3$
$f'''(0) = 7$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$f(x) = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + O(x^4)$$

Tip: The alternative option here would be to substitute into the known expansion for e^x.

Using the $O(x^n)$ notation, we can avoid unnecessary calculation of irrelevant terms. Anything that would result in a power of x greater than 3 can simply be ignored, captured by $O(x^4)$:

$$e^{x+x^2} = 1 + (x + x^2) + \frac{1}{2}(x + x^2)^2 + \frac{1}{6}(x + x^2)^3 + O(x)^4$$
$$= 1 + x + x^2 + \frac{1}{2}x^2 + x^3 + \frac{1}{6}x^3 + O(x^4)$$
$$= 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + O(x^4)$$

20

$$f(x) = \sqrt{\cos x}$$
$$f'(x) = -\frac{1}{2}\frac{\sin x}{\sqrt{\cos x}}$$
$$f''(x) = -\frac{\frac{1}{2}\left(\cos x\sqrt{\cos x} - \frac{\frac{1}{2}\sin^2 x}{\sqrt{\cos x}}\right)}{\cos x}$$
$$f(0) = 1$$
$$f'(0) = 0$$
$$f''(0) = -\frac{1}{2}$$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + O(x^3)$$
$$f(x) = 1 - \frac{1}{4}x^2 + O(x^3)$$

21 **a**

Maclaurin series for $f(x) = \ln(1 + x)$ is $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$

So Maclaurin series for $g(x) = \ln(1 - 3x)$ is $g(x) = -3x - \frac{9}{2}x^2 - 9x^3 + O(x^4)$

b

$$\frac{\ln(1 - 3x)}{2x} = -\frac{3}{2} + O(x)$$

So $\lim_{x\to 0}\frac{\ln(1 - 3x)}{2x} = -\frac{3}{2}$

22

Maclaurin series for $f(x) = \sin 2x$ is

$$2x - \frac{(2x)^3}{3!} + O(x^5)$$

Then $\frac{\sin 2x}{x} = 2 - O(x^2)$

$$\lim_{x\to 0}\frac{\sin 2x}{x} = 2$$

23 **a**

$\mathrm{f}(x) = \tan x$
$\mathrm{f}'(x) = \sec^2 x$
$\mathrm{f}''(x) = 2\sec^2 x \tan x$
$\mathrm{f}'''(x) = 4\sec^2 x \tan x + 2\sec^4 x$
$\mathrm{f}^{(4)}(x) = 8\sec^2 \tan x + 8\sec^4 \tan x$
$\mathrm{f}^{(5)}(x) = 16\sec^2 \tan x + 8\sec^4 x + 32\sec^4 \tan x + 8\sec^6 x$
$\mathrm{f}(0) = 0$
$\mathrm{f}'(0) = 1$
$\mathrm{f}''(0) = 0$
$\mathrm{f}'''(0) = 2$
$\mathrm{f}^{(4)}(0) = 0$
$\mathrm{f}^{(5)}(0) = 16$

Maclaurin series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \frac{x^4}{4!}\mathrm{f}^{(4)}(0) + \frac{x^5}{5!}\mathrm{f}^{(5)}(0) + O(x^6)$$

$$\mathrm{f}(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6)$$

b

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)$$

$$\begin{aligned}\mathrm{e}^x \tan x &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)\right)\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6)\right)\\ &= x + x^2 + \left(\frac{1}{2} + \frac{1}{3}\right)x^3 + \left(\frac{1}{6} + \frac{1}{3}\right)x^4 + O(x^5)\\ &= x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + O(x^5)\end{aligned}$$

24 **a**

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)$$

$$\cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

$$\begin{aligned}\sin x \cos x &= \left(x - \frac{1}{6}x^3 + O(x^5)\right)\left(1 - \frac{1}{2}x^2 + O(x^4)\right)\\ &= x + x^3\left(-\frac{1}{2} - \frac{1}{6}\right) + O(x^5)\\ &= x - \frac{2}{3}x^3 + O(x^5)\end{aligned}$$

b

$$\begin{aligned}\sin x \cos x &= \frac{1}{2}\sin 2x\\ &= \frac{1}{2}\left[(2x) - \frac{1}{6}(2x)^3 + O(x^5)\right]\\ &= \frac{1}{2}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]\\ &= x - \frac{2}{3}x^3 + O(x^5)\end{aligned}$$

25 **a**

$$\ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}(\ln(1+x) - \ln(1-x))$$

$$= \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \left((-x) - \frac{1}{2}(-x)^2 + \frac{1}{3}(-x)^3 - \frac{1}{4}(-x)^4\right) + O(x^5)\right)$$

$$= \frac{1}{2}\left(2x + \frac{2}{3}x^3 + O(x^5)\right)$$

$$= x + \frac{1}{3}x^3 + O(x^5)$$

b

When $x = \frac{4}{5}$, $\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1.8}{0.2}} = \sqrt{9} = 3$

Substituting:

$$\ln 3 \approx \frac{4}{5} + \frac{1}{3}\left(\frac{64}{125}\right)$$

$$\approx \frac{364}{375}$$

Tip: This is not a unique, or even a very good, approximation; the smaller the value of x, the more accurate the truncated series will be. $x = 0.5$ gives $\ln\sqrt{\frac{1+x}{1-x}} = \ln\sqrt{3} = \frac{1}{2}\ln 3$.

Then $\ln 3 \approx 2\left(\frac{1}{2} + \frac{1}{24}\right) = \frac{13}{12}$

This is a more accurate approximation (1.3% error instead of 11.6%).

As long as you give clear justification for your decision, any relevant and useful answer will be acceptable.

26 **a**

$$\sin 2x = (2x) - \frac{1}{6}(2x)^3 + O(x^5)$$

$$= 2x - \frac{4}{3}x^3 + O(x^5)$$

b

$$\sin 2x = 7x^3$$

$$2x \approx \frac{25}{3}x^3$$

$$x^2 \approx \frac{6}{25}$$

$$x \approx \frac{\sqrt{6}}{5}$$

Worked solutions

27 **a**

$f(x) = \arcsin x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-0.5}$

$f''(x) = x(1-x^2)^{-1.5}$

$f'''(x) = (1-x^2)^{-1.5} + 3x^2(1-x^2)^{-2.5}$

b

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = 1$

Maclaurin series:

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$f(x) = x + \frac{1}{6}x^3 + O(x^4)$

c

$\sin 2x = (2x) - \frac{1}{6}(2x)^3 + O(x^5)$

$= 2x - \frac{4}{3}x^3 + O(x^5)$

d

$\sin 2x = \arcsin x$

$2x - \frac{4}{3}x^3 \approx x + \frac{1}{6}x^3$

$x \approx \frac{9}{6}x^3$

$x^2 \approx \frac{2}{3}$

$x \approx \sqrt{\frac{2}{3}}$

28 **a**

$\ln(e+x) = \ln e + \ln\left(1+\frac{x}{e}\right)$

$= 1 + \left(\frac{x}{e}\right) - \frac{1}{2}\left(\frac{x}{e}\right)^2 + \frac{1}{3}\left(\frac{x}{e}\right)^3 + O(x^4)$

$= 1 + \frac{x}{e} - \frac{1}{2e^2}x^2 + \frac{1}{3e^3}x^3 + O(x^4)$

b

When $x = e$:

$\ln 2e \approx 1 + 1 - \frac{1}{2} + \frac{1}{3} = 1 + \frac{5}{6}$

So $\ln e + \ln 2 = 1 + \frac{5}{6}$

$\ln 2 \approx \frac{5}{6}$

Note: The convergence for the series of $\ln\left(1+\frac{x}{e}\right)$ is only valid for $-e < x \leq e$ so we are approximating at the boundary of convergence.

29 **a**

$$e^{-x^2} = 1 + (-x^2) + \frac{1}{2}(-x^2) + O(x^6)$$
$$= 1 - x^2 + \frac{1}{2}x^4 + O(x^6)$$

b

$$\int_0^1 e^{-x^2}\,dx \approx \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_0^1$$
$$\approx \frac{23}{30}$$

c

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$
$$= 2.66\%$$

d

$$\int_0^{0,1} e^{-x^2}\,dx \approx \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_0^{0.1}$$
$$\approx \frac{29903}{300000}$$

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$
$$= 0.009\%$$

The accuracy is far greater for the smaller value of x.

30 **a**

$f(x) = e^{-x}\tan x$
$f'(x) = e^{-x}(\sec^2 x - \tan x)$
$f''(x) = e^{-x}(2\sec^2 x \tan x - 2\sec^2 x + \tan x)$
$f'''(x) = e^{-x}(4\sec^2 x \tan^2 x + 2\sec^4 x - 6\sec^2 x \tan x + 3\sec^2 x - \tan x)$
$f(0) = 0$
$f'(0) = 1$
$f''(0) = -2$
$f'''(0) = 5$
Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$
$$f(x) = x - x^2 + \frac{5}{6}x^3 + O(x^4)$$

b

$$\int_0^1 e^{-x}\tan x\,dx \approx \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4\right]_0^1$$
$$\approx \frac{3}{8}$$

c

$$\frac{e^{-x}\tan x}{2x} = \frac{1}{2} + O(x)$$

So $\lim_{x\to 0}\frac{e^{-x}\tan x}{2x} = \frac{1}{2}$

31 **a**

$f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-0.5}$

$f'(x) = x(1-x^2)^{-1.5}$

$f''(x) = (1-x^2)^{-1.5} + 3x^2(1-x^2)^{-2.5}$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = 1$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + O(x^3)$$

$$f(x) = 1 + \frac{1}{2}x^2 + O(x^3)$$

b

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-u^2}}\,du = \left[u + \frac{1}{6}u^3 + O(u^5)\right]_0^x = x + \frac{1}{6}x^3 + O(x^5)$$

c

Let $x = \frac{1}{2}$

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \approx \frac{1}{2} + \frac{1}{48} = \frac{25}{48}$$

So $\pi \approx \frac{25}{8}$

32 **a**

Maclaurin series for e^{-2x}:

$$e^{-2x} = 1 + (-2x) + \frac{1}{2}(-2x)^2 + \frac{1}{6}(-2x)^3 + O(x^4)$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + O(x^4)$$

So

$$(1+x)e^{-2x} = (1+x)\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + O(x^4)\right)$$

$$= 1 - x + \frac{2}{3}x^3 + O(x^4)$$

b

$$\int_0^1 (1+x)e^{-2x}\,dx \approx \left[x - \frac{1}{2}x^2 + \frac{1}{6}x^4\right]_0^1$$

$$\approx \frac{2}{3}$$

33 **a**

$$x\cos x = x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\right)$$

$$= x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + O(x^7)$$

b

$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)$

$\frac{\sin x - x\cos x}{x^3} = \frac{1}{3} - \frac{1}{30}x^2 + O(x^4)$

$\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3} = \frac{1}{3}$

34

$x - \sin x = x - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)\right) = \frac{1}{6}x^3 - \frac{1}{120}x^5 + O(x^7)$

$\mathrm{f}(x) = \tan x$
$\mathrm{f}'(x) = \sec^2 x$
$\mathrm{f}''(x) = 2\sec^2 x \tan x$
$\mathrm{f}'''(x) = 2\sec^4 x + 4\sec^2 x\tan^2 x$
$\mathrm{f}(0) = 0$
$\mathrm{f}'(0) = 1$
$\mathrm{f}''(0) = 0$
$\mathrm{f}'''(0) = 2$
Maclaurin series:

$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + O(x^4)$

$\mathrm{f}(x) = x + \frac{1}{3}x^3 + O(x^4)$

$x - \tan x = -\frac{1}{3}x^3 + O(x^4)$

So $\frac{x - \sin x}{x - \tan x} = \frac{\frac{1}{6}x^3 + O(x^5)}{-\frac{1}{3}x^3 + O(x^4)} = \frac{\frac{1}{6} + O(x^2)}{-\frac{1}{3} + O(x)} = \frac{1 + O(x^2)}{-2 + O(x)}$

$\lim_{x\to 0}\frac{x - \sin x}{x - \tan x} = -\frac{1}{2}$

35 **a**

$$\begin{aligned}\ln(1 + x^2) &= (x^2) - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \frac{1}{4}(x^2)^4 + O(x^{10})\\ &= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + O(x^{10})\end{aligned}$$

b

$1 - \cos x = 1 - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\right) = \frac{1}{2}x^2 - \frac{1}{24}x^4 + O(x^6)$

$\frac{\ln(1 + x^2)}{1 - \cos x} = \frac{x^2 + O(x^4)}{\frac{1}{2}x^2 + O(x^4)} = \frac{2 + O(x^2)}{1 + O(x^2)}$

$\lim_{x\to 0}\frac{\ln(1 + x^2)}{1 - \cos x} = 2$

36

Using standard Maclaurin series:

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4)$$

Using binomial expansion:

$$(1+x)^{-0.5} = 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + O(x^4)$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + O(x^4)$$

$$e^{-x}(1+x)^{-0.5} = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4)\right)\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + O(x^4)\right)$$

$$= 1 + x\left(-1 - \frac{1}{2}\right) + x^2\left(\frac{1}{2} + \frac{1}{2} + \frac{3}{8}\right) + O(x^3)$$

$$= 1 - \frac{3}{2}x + \frac{11}{8}x^2 + O(x^3)$$

37 **a**

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

$$\ln(2+x) = \ln 2 + \ln\left(1 + \frac{x}{2}\right)$$

$$= \ln 2 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{24}x^3 + O(x^4)$$

b

$$\ln[(2+x)^2(1-x)^3] = 2\ln(2+x) + 3\ln(1-x)$$

$$= 2\left(\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3\right) + 3\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right) + O(x^4)$$

$$= 2\ln 2 - 2x - \frac{7}{4}x^2 - \frac{11}{12}x^3 + O(x^4)$$

38 **a**

$$x\sin x = x\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)\right)$$

$$= x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + O(x^8)$$

b

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x\sin x}$$

$$= \frac{\frac{1}{6}x^3 - \frac{1}{120}x^5 + O(x^7)}{x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + O(x^8)}$$

$$= \frac{x + O(x^3)}{6 + O(x^2)}$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$$

39 **a**

$$x\ln(1+x) = x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)\right)$$
$$= x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{4}x^5 + O(x^6)$$

b

$$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{x - \ln(1+x)}{x\ln(1+x)}$$
$$= \frac{\frac{1}{2}x^2 + O(x^3)}{x^2 + O(x^3)}$$
$$= \frac{1 + O(x^1)}{2 + O(x^1)}$$
$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right) = \frac{1}{2}$$

40 **a**

$$2^x = \mathrm{e}^{x\ln 2}$$

b

$$2^x = 1 + (x\ln 2) + \frac{1}{2}(x\ln 2)^2 + O(x^3)$$
$$= 1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2 + O(x^3)$$

c

$$\frac{2^x - 1}{3^x - 1} = \frac{(\ln 2)x + O(x^2)}{(\ln 3)x + O(x^2)}$$
$$= \frac{\ln 2 + O(x^1)}{\ln 3 + O(x^1)}$$

$$\lim_{x\to 0}\left(\frac{2^x - 1}{3^x - 1}\right) = \frac{\ln 2}{\ln 3} = \log_3 2$$

41 **a**

$$\mathrm{f}(x) = \ln(\mathrm{e}^x\cos x)$$
$$\mathrm{f}'(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x\cos x)}{\mathrm{e}^x\cos x} = \frac{\mathrm{e}^x(\cos x - \sin x)}{\mathrm{e}^x\cos x} = 1 - \tan x$$

$$\mathrm{f}''(x) = -\sec^2 x$$
$$\mathrm{f}'''(x) = -2\sec^2 x\tan x$$
$$\mathrm{f}^{(4)}(x) = -4\sec^2 x\tan^2 x - 2\sec^4 x$$

b)

$$\mathrm{f}(0) = 0$$
$$\mathrm{f}'(0) = 1$$
$$\mathrm{f}''(0) = -1$$
$$\mathrm{f}'''(0) = 0$$
$$\mathrm{f}^{(4)}(0) = -2$$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + O(x^5)$$

$$f(x) = x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + O(x^5)$$

$$\frac{1}{x^3}\ln((1+x)^{-1}e^x \cos x) = \frac{1}{x^3}[-\ln(1+x) + \ln(e^x \cos x)]$$

$$= \frac{1}{x^3}\left[\left(-x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4\right) + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4\right) + O(x^5)\right]$$

$$= -\frac{1}{3} + \frac{1}{6}x + O(x^2)$$

$$\lim_{x\to 0}\frac{1}{x^3}\ln\left(\frac{e^x \cos x}{1+x}\right) = -\frac{1}{3}$$

42 **a**

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)$$

b

Substituting into $e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \frac{1}{24}a^4 + O(a^5)$:

$$e^{\sin x} = 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 + \frac{1}{24}\left(x - \frac{1}{6}x^3\right)^4 + O(x^5)$$

$$= 1 + x + \frac{1}{2}x^2 + x^3\left(-\frac{1}{6} + \frac{1}{6}\right) + x^4\left(-\frac{1}{6} + \frac{1}{24}\right) + O(x^5)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^5)$$

43 **ai**

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + O(x^6)$$

aii

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7)$$

b

$\arctan(\ln(1+x))$

$$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5\right) - \frac{1}{3}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)^3 + \frac{1}{5}(x)^5 + O(x^6)$$

Coefficient of

x^1: 1

x^2: $-\frac{1}{2}$

x^3: $\frac{1}{3} - \frac{1}{3} = 0$

x^4: $-\frac{1}{4} + 3\left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

x^5: $\frac{1}{5} + 3\left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) + \frac{1}{5} = \frac{1}{5} - \frac{1}{4} - \frac{1}{3} + \frac{1}{5} = -\frac{11}{60}$

$$\arctan(\ln(1+x)) = x - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{11}{60}x^5 + O(x^6)$$

44 **a** $\frac{(-1)^n}{(2n+1)!}$

b $\frac{(-1)^n}{(2n+1)!}$

c

The coefficient of x^{2n} in the series for $\cos x$ is $\frac{(-1)^n}{(2n)!}$

$$\frac{\sin x}{x} - \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n} - \frac{(-1)^n}{(2n)!}x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{-1-(2n+1)}{(2n+1)!}(-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} -\frac{2n}{(2n+1)!}(-1)^n x^{2n}$$

$$= 0x^0 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2n}{(2n+1)!}x^{2n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2n}{(2n+1)!}x^{2n}$$

45 **a** $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$

Substituting:

$$\mathrm{e}^{2x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(2x^2)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^{2n}$$

The coefficient of x^{2n} is $\frac{2^n}{n!}$

b

The coefficient of x^{2n} in the expansion of $2\cos x$ is $2\frac{(-1)^n}{(2n)!}$

The coefficient of x^{2n} in the expansion of $\mathrm{e}^{2x^2} - 2\cos x$ is $\frac{2^n}{n!} - 2\frac{(-1)^n}{(2n)!}$

46 **a** $\frac{(-1)^{n+1}}{n}$ for $n \in \mathbb{Z}^+$

b

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}x^n - \frac{(-1)^{n+1}}{n}(-x)^n\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}x^n + \frac{1}{n}x^n\right)$$

$$= \sum_{\text{odd } n} \frac{2}{n}x^n$$

$$= \sum_{n=0}^{\infty} \frac{2}{2n+1}x^{2n+1}$$

47

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + O(x^3)$$

$$= 3 - \frac{x}{4} + \frac{3x^2}{2} + O(x^3)$$

So $f'(0) = -\frac{1}{4}, f(0) = 3$

The tangent at $x = 0$ is therefore

$$y - 3 = -\frac{1}{4}(x - 0)$$

$$y = -\frac{1}{4}x + 3$$

48

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$= 2 - 4x^2 + \frac{3}{2}x^3 + O(x^4)$$

So $f'(0) = 0$, and there is therefore a stationary point at $x = 0$

$f(0) = 2, f''(0) = -8 < 0$

Stationary point (0,2) is a local maximum.

49

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + O(x^3)$$

$$= \sum_0^\infty \frac{3n+2}{4(2n+1)!}x^n$$

$$= \frac{1}{2} + \frac{5}{24}x + \frac{1}{60}x^2 + O(x^3)$$

a $f(0) = \frac{1}{2}$

b $f''(0) = \frac{1}{30}$

50 **a**

$f(x) = xe^x$

Proposition: $f^{(n)}(x) = (n + x)e^x$

Base case $n = 0$: $f^{(0)}(x) = f(x) = (0 + x)e^x$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $f^{(k)}(x) = (k + x)e^x$

Working towards: $f^{(k+1)}(x) = (k + 1 + x)e^x$

$$f^{(k+1)}(x) = \frac{d}{dx}[(k + x)e^x] \quad \text{(by assumption)}$$

$$= (k + x)e^x + e^x$$

$$= (k + 1 + x)e^x$$

So the proposition is true for $n = k + 1$

Conclusion: The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 2$
$f'''(0) = 3$
$f^{(4)}(0) = 4$
$f^{(5)}(0) = 5$
Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + O(x^6)$$

$$f(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + O(x^6)$$

Exercise 10E

7

$y'' - 2y' + y = 0$
Let $y = a_0 + a_1x + a_2x^2 + O(x^3)$
$y(0) = a_0 = 1$
$y'(0) = a_1 = 1$
$y''(0) = 2a_2 = 2y'(0) - y(0) = 1$

$$a_2 = \frac{1}{2}$$

$$y = 1 + x + \frac{1}{2}x^2 + O(x^3)$$

8 **a**

$y'' - 3y' + 2y = 0$
Let $y = a_0 + a_1x + a_2x^2 + O(x^3)$
$y(0) = a_0 = 1$
$y'(0) = a_1 = 2$
$y''(0) = 2a_2 = 3y'(0) - 2y(0) = 4$
$a_2 = 2$
$y = 1 + 2x + 2x^2 + O(x^3)$

b $y(1) \approx 5$

9 **a**

Using binomial expansion:

$$(1 + x^3)^{0.5} = 1 + \frac{1}{2}(x^3) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^3)^2 + O(x^9)$$
$$= 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + O(x^9)$$

b

$$y = \int_0^x (1 + u^3)^{0.5}\ du$$
$$= c + x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + O(x^{10})$$

c

$y(0) = 1 = c$

$y(0.1) \approx 1 + 0.1 + 0.0000125 - \dfrac{1}{560000000}$

≈ 1.10001

10

$y''' + y = 0$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$

$y(0) = a_0 = 1$

$y'(0) = a_1 = 2$

$y''(0) = 4 = 2a_2$ so $a_2 = 2$

$y'''(0) = 6a_3 = -y(0) = -1$

$a_3 = -\dfrac{1}{6}$

$y = 1 + 2x + 2x^2 - \dfrac{1}{6}x^3 + O(x^4)$

11 **a**

$y'' + xy = 0$ so $y''' + xy' + y = 0$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$

$y(0) = a_0 = 1$

$y'(0) = a_1 = 0$

$y''(0) = 2a_2 = -0 \times y(0) = 0$ so $a_2 = 0$

$y'''(0) = 6a_3 = -0 \times y'(0) - y(0) = -1$

$a_3 = -\dfrac{1}{6}$

$y = 1 - \dfrac{1}{6}x^3 + O(x^4)$

b $y(0.5) \approx 0.979$

12

$y'' + y^2 = 0$ so $y''' = -2yy'$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$

$y(0) = a_0 = 1$

$y'(0) = a_1 = -1$

$y''(0) = 2a_2 = -\left(y(0)\right)^2 = -1$ so $a_2 = -\dfrac{1}{2}$

$y'''(0) = 6a_3 = -2y(0) \times y'(0) = 2$

$a_3 = \dfrac{1}{3}$

$y = 1 - x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + O(x^4)$

13

$\ddot{y} + (\dot{y})^2 + y^2 = t$

$\dddot{y} + 2\dot{y}\ddot{y} + 2y\dot{y} = 1$

Let $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + O(t^4)$

$y(0) = a_0 = -2$

$\dot{y}(0) = a_1 = 3$

$\ddot{y}(0) = 2a_2 = 0 - \left(\dot{y}(0)\right)^2 - \left(y(0)\right)^2 = -13 \text{ so } a_2 = -\frac{13}{2}$

$\dddot{y}(0) = 6a_3 = 1 - 2\dot{y}(0) \times \ddot{y}(0) - 2y(0) \times \dot{y}(0) = 1 + 78 + 12 = 91$

$a_3 = \frac{91}{6}$

$y = -2 + 3t - \frac{13}{2}t^2 + \frac{91}{6}t^3 + O(t^4)$

14 **a**

$$xe^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1}$$

b

$$\frac{dy}{dx} = xe^x$$

$$y = c + \sum_{k=0}^{\infty} \frac{1}{(k+2) \times k!} x^{k+2}$$

c

$y(0) = c = 1$

$y = 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$

$y(0.5) \approx 1.17$

15

$y' + e^y = \cos x$

$y'' + e^y y' = -\sin x$

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4)$

$y(0) = a_0 = 1$

$y'(0) = a_1 = \cos(0) - e^{y(0)} = 1 - e$

$y''(0) = 2a_2 = -\sin(0) - e^{y(0)}y'(0) = 0 - e(1 - e) \text{ so } a_2 = \frac{1}{2}(e^2 - e)$

$y = 1 + (1 - e)x + \frac{1}{2}(e^2 - e)x^2 + O(x^3)$

16 **a**

$y'' - 4y' + 4y = 0$

Let $y = \sum_{k=0}^{\infty} a_k x^k$

Then $y' = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k$

and $y'' = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k$

Substituting the general coefficient for x^k into the differential equation:

$(k+1)(k+2)a_{k+2} - 4(k+1)a_{k+1} + 4a_k = 0$ for every $k \geq 0$

$$a_{k+2} = \frac{4(k+1)a_{k+1} - 4a_k}{(k+2)(k+1)}$$

b

If $y = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0} \frac{2^{k+1}x^{k+1}}{k!}$ then $a_0 = 0, a_k = \frac{2^k}{(k-1)!}$ for $k \geq 1$

Using the iteration from part **a**: $a_2 = \frac{4a_1 - 4a_0}{2} = 4$

Proposition: $a_n = \frac{2^n}{(n-1)!}$ for $n \geq 1$

Base case $n = 1$: $a_1 = 2 = \frac{2^1}{0!}$

Base case $n = 2$: $a_2 = 4 = \frac{2^2}{1!}$

Inductive step: Assume the proposition is true for integer $n = k-1$ and $n = k \geq 2$

So $a_{k-1} = \frac{2^{k-1}}{(k-2)!}$ and $a_k = \frac{2^k}{(k-1)!}$

Working towards: $a_{k+1} = \frac{2^{k+1}}{k!}$

$$a_{k+1} = \frac{4ka_k - 4a_{k-1}}{k(k+1)} \quad \text{(by assumption and iteration)}$$

$$= \frac{k \times 2^{k+2} - (k-1)2^{k+1}}{k(k+1)(k-1)!}$$

$$= \frac{2^{k+1}}{(k+1)!}\big(2k - (k-1)\big)$$

$$= \frac{2^{k+1}(k+1)}{(k+1)!}$$

$$= \frac{2^{k+1}}{k!}$$

So the proposition is true for $n = k+1$

Conclus The proposition is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

17 **a**

$y'' - xy' - y = 0$

Let $y = \sum_{k=0}^{\infty} a_k x^k$

Then $y' = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k$ so $xy' = \sum_{k=1}^{\infty} k a_k x^k$

and $y'' = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k$

Substituting the general coefficient for x^k into the differential equation:

$(k+1)(k+2)a_{k+2} - ka_k - a_k = 0$ for every $k \geq 0$

$a_{k+2} = \frac{a_k}{(k+2)}$

$y(0) = a_0 = 1$

$y'(0) = a_1 = 0$

Then by the iterative formula, all odd coefficients will equal zero.

$a_{2k} = \frac{a_{2(k-1)}}{2k}$

Considering instead as a series of powers of x^2:

Let $y = \sum_{k=0}^{\infty} b_k x^{2k}$ where $b_k = a_{2k}$

$b_k = \frac{b_{k-1}}{2k}$ and $b_0 = 1$ so $b_k = \frac{1}{2^k k!}$

$y = \sum_{k=0} \frac{x^{2k}}{2^k k!}$

b

$y = \sum_{k=0} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = \mathrm{e}^{\frac{x^2}{2}}$

18

$y'' + x^2y' + xy = 0$

Let $y = \sum_{k=0}^{\infty} a_k x^k$ so $xy = \sum_{k=1}^{\infty} a_{k-1}x^k$

Then $y' = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k$ so $x^2y' = \sum_{k=2}^{\infty} (k-1)a_{k-1}x^k$

and $y'' = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k$

Substituting the general coefficient for x^k into the differential equation:

$(k+1)(k+2)a_{k+2} + (k-1)a_{k-1} + a_{k-1} = 0$ for every $k \geq 2$

$a_{k+2} = -\frac{ka_{k-1}}{(k+1)(k+2)}$

$y(0) = a_0 = 0$

$y'(0) = a_1 = 1$
$y''(0) = 2a_2 = 0$
The recurrence relation on the coefficients means that $a_{3k} = a_{3k+2} = 0$ for integer k; only the a_{3k+1} coefficients will be non-zero.
$a_1 = 1$
$a_4 = -\frac{2}{3 \times 4}$
$a_7 = \frac{2 \times 5}{3 \times 4 \times 6 \times 7} \dots$
$$a_{3k+1} = (-1)^k \frac{2 \times 5 \times \dots \times (3k-1)}{3 \times 4 \times 6 \times 7 \times \dots \times (3k) \times (3k+1)}$$
$$= (-1)^k \frac{2^2 \times 5^2 \times \dots \times (3k-1)^2}{(3k+1)!}$$
$$y = x + \sum_{k=1}^{\infty} (-1)^k \frac{2^2 \times 5^2 \times \dots \times (3k-1)^2}{(3k+1)!} x^{3k+1}$$

19
$y'' + xy' + y = 0$
Let $y = \sum_{k=0}^{\infty} a_k x^k$
Then $y' = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k$ so $xy' = \sum_{k=0}^{\infty} ka_k x^k$
and $y'' = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k$
Substituting the general coefficient for x^k into the differential equation:
$(k+1)(k+2)a_{k+2} + ka_k + a_k = 0$ for every $k \geq 0$
$a_{k+2} = -\frac{a_k}{(k+2)}$
$y(0) = a_0$
$y'(0) = a_1$
The recurrence relation links all odd coefficients in terms of a_1 and all even coefficients in terms of a_0.
Even coefficients:
$a_2 = a_0\left(-\frac{1}{2}\right)$
$a_4 = a_0\left(\frac{1}{2 \times 4}\right) = a_0\left(\frac{1}{2^2} \times \frac{1}{2!}\right)$
$a_{2k} = \frac{a_0(-1)^k}{2^k k!}$
Odd coefficients:
$a_3 = a_1\left(-\frac{1}{3}\right)$
$a_5 = a_1\left(\frac{1}{3 \times 5}\right) = a_1\left(\frac{2 \times 4}{5!}\right) = a_1\left(\frac{2^2 \times 2!}{5!}\right)$
$a_7 = a_1\left(\frac{-1}{3 \times 5 \times 7}\right) = -a_1\left(\frac{2 \times 4 \times 6}{7!}\right) = -a_1\left(\frac{2^3 \times 3!}{7!}\right)$

$$a_{2k+1} = \frac{a_1(-1)^k 2^k k!}{(2k+1)!}$$

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-2)^k k!}{(2k+1)!} x^{2k+1}$$

20 **a**

$(1 - x^2)y'' - 2xy' + l(l+1)y = 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$

Then $y' = \sum_{r=0}^{\infty} (r+1)a_{r+1}x^r$ so $xy' = \sum_{r=0}^{\infty} r a_r x^r$

and $y'' = \sum_{r=0}^{\infty} (r+1)(r+2)a_{r+2}x^r$ so $(1-x^2)y''$

$$= \sum_{r=0}^{\infty} [(r+1)(r+2)a_{r+1} - r(r-1)a_r]x^r$$

Substituting the general coefficient for x^r into the differential equation:

$(r+1)(r+2)a_{r+2} - r(r-1)a_r - 2ra_r + l(l+1)a_r = 0$ for every $r \geq 0$

$(r+1)(r+2)a_{r+2} = a_r\left(r^2 + r - l(l+1)\right)$

$$a_{r+2} = \frac{r(r+1) - l(l+1)}{(r+1)(r+2)} a_r$$

b

$$y(1) = \sum_{r=0}^{\infty} a_r = 1$$

The relationship given in part **a** means that when $r = l$, $a_{r+2} = 0$ and therefore $a_{r+2n} = 0$.

For finite series, since there is no restriction on the alternating values, the odd or even coefficients not forced to zero by the relationship must therefore be zero throughout,

bi If $l = 1$ then even coefficients $a_{2r} = 0$

$a_3 = 0 = a_{1+2r}$

So $y = a_1 x$

$y(1) = 1 = a_1$

So $y = x$

bii If $l = 2$ then odd coefficients $a_{2r+1} = 0$

$a_4 = 0 = a_{2+2r}$

So $y = a_0 + a_2 x^2$

$y(1) = 1 = a_0 + a_2$

$$a_2 = \frac{0^2 + 0 - 2(2+1)}{(0+1)(0+2)} a_0 = -3a_0$$

$a_0 + a_2 = -2a_0 = 1$

$$a_0 = -\frac{1}{2}$$

$$y = \frac{1}{2}(3x^2 - 1)$$

21 **a**

$y'' - 2xy' + 2ky = 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$

Then $y' = \sum_{r=0}^{\infty} (r+1)a_{r+1}x^r$ so $xy' = \sum_{r=0}^{\infty} r a_r x^r$

and $y'' = \sum_{r=0}^{\infty} (r+1)(r+2)a_{r+2}x^r$

Substituting the general coefficient for x^r into the differential equation:

$(r+1)(r+2)a_{r+2} - 2ra_r + 2ka_r = 0$ for every $r \geq 0$

$(r+1)(r+2)a_{r+2} = 2a_r(r-k)$

$a_{r+2} = \frac{2(r-k)}{(r+1)(r+2)} a_r$

b

$y(0) = a_0 = 0$

$y'(0) = a_1 = 1$

$a_{2r} = 0$ for all integer r because of the relationship in part **a**

For the solution to be a finite polynomial, it is necessary that $r - k = 0$ for some odd positive integer r, so the requirement on k is that it must be a positive, odd integer.

For $k = 1, a_3 = 0$ so $y = x$

For $k = 3, a_3 = -\frac{2}{3}$ and $a_5 = 0$ so $y = x - \frac{2}{3}x^3$

22 **a**

$(1 - x^2)\mathrm{f}''(x) - x\mathrm{f}'(x) + k^2\mathrm{f}(x) = 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$

Then $y' = \sum_{r=0}^{\infty} (r+1)a_{r+1}x^r$ so $xy' = \sum_{r=0}^{\infty} r a_r x^r$

and $y'' = \sum_{r=0}^{\infty} (r+1)(r+2)a_{r+2}x^r$ so $(1 - x^2)y''$

$$= \sum_{r=0}^{\infty} [(r+1)(r+2)a_{r+1} - r(r-1)a_r]x^r$$

Substituting the general coefficient for x^r into the differential equation:

$(r+1)(r+2)a_{r+2} - r(r-1)a_r - ra_r + k^2 a_r = 0$ for every $r \geq 0$

$(r+1)(r+2)a_{r+2} = a_r(r^2 - k^2)$

$a_{r+2} = \frac{r^2 - k^2}{(r+1)(r+2)} a_r$

b

The relationship between coefficients in part **a** means that for positive integer k, all coefficients a_{k+2n} will be zero.
For a finite polynomial, if k is odd then all even coefficients a_{2n} must be zero and if k is even then all odd coefficients a_{2n+1} must be zero.
The degree of the finite polynomial will be k, since a_k will be the final non-zero coefficient.
$k = 2$: quadratic Chebyschev function:

$$a_2 = -\frac{4}{(1)(2)}a_0 = -2a_0$$

$$y = a_0(1 - 2x^2)$$

Require that the range for $-1 \leq x \leq 1$ is $-1 \leq y \leq 1$ so $a_0 = \pm 1$

$$y = \pm(1 - 2x^2)$$

Tip: Strictly, the Chebyschev polynomial will be the polynomial with greatest lead coefficient to satisfy the conditions, so $y = 2x^2 - 1$ is the unique correct solution.

$k = 3$: cubic Chebyschev function

$$a_3 = \frac{1-9}{(2)(3)}a_1 = -\frac{4}{3}a_1$$

$$y = a_1\left(x - \frac{4}{3}x^3\right)$$

Require that the range for $-1 \leq x \leq 1$ is $-1 \leq y \leq 1$ so $a_1 = \pm 3$

$$y = \pm(3x - 4x^3)$$

Tip: As above, $y = 4x^3 - 3x$ is the unique correct solution.

23 a

$$4x^2y'' + 4xy' - y = 0$$

Let $y = \sum_{r=0}^{\infty} a_r x^r$

Then $y' = \sum_{r=0}^{\infty}(r+1)a_{r+1}x^r$ so $xy' = \sum_{r=0}^{\infty} ra_r x^r$

and $y'' = \sum_{r=0}^{\infty}(r+1)(r+2)a_{r+2}x^r$ so $x^2y'' = \sum_{r=0}^{\infty} r(r-1)a_r x^r$

Substituting the general coefficient for x^r into the differential equation:
$4r(r-1)a_r + 4ra_r - a_r = 0$ for every $r \geq 0$
$(4r^2 - 1)a_r = 0$
Since $4r^2 - 1 \neq 0$ for any $r \in \mathbb{N}$, it follows that $a_r = 0$ for all a_r.
That is, the only Maclaurin series solution is $y = 0$.

b

Let $y = \sum_{r=0}^{\infty} a_r x^{r+m}$

Then $y' = \sum_{r=0}^{\infty}(r+1)a_{r+1}x^{r+m}$ so $xy' = \sum_{r=0}^{\infty}(r+m)a_r x^r$

and $y'' = \sum_{r=0}^{\infty}(r+1)(r+2)a_{r+2}x^r$ so $x^2y'' = \sum_{r=0}^{\infty}(r+m)(r+m-1)a_r x^{r+m}$

Substituting the general coefficient for x^r into the differential equation:

$4(r+m)(r+m-1)a_r + 4(r+m)a_r - a_r = 0$ for every $r \geq 0$

$(4(r+m)^2 - 1)a_r = 0$

Then $a_r = 0$ except when $(r+m)^2 = \frac{1}{4}$

$r + m = \pm\frac{1}{2}$

$y = \frac{A}{\sqrt{x}} + B\sqrt{x}$

Mixed Practice

1 **a**

$\int y^{-1}\ \mathrm{d}y = \int 3\cos 2x\ \mathrm{d}x$

$\ln|y| = 1.5\sin 2x + c$

$|y| = A\mathrm{e}^{1.5\sin 2x}$ for $A > 0$

$y = A\mathrm{e}^{1.5\sin 2x}$

b

$y(0) = 5 = A$

$y = 5\mathrm{e}^{1.5\sin 2x}$

2

$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x^2 - 2)y$

$\int y^{-1}\ \mathrm{d}y = \int 3x^2 - 2\ \mathrm{d}x$

$\ln|y| = x^3 - 2x + c$

$|y| = A\mathrm{e}^{x^3-2x}$ for $A > 0$

$y = A\mathrm{e}^{x^3-2x}$

3 **a**

$\mu = \mathrm{e}^{\int -3x^2\,\mathrm{d}x} = \mathrm{e}^{-x^3}$

b

Multiplying both sides by the integrating factor:

$\mathrm{e}^{-x^3}\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2\mathrm{e}^{-x^3}y = 6x^2\mathrm{e}^{-x^3}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^3}y\right) = 6x^2\mathrm{e}^{-x^3}$

$\mathrm{e}^{-x^3}y = -2\mathrm{e}^{-x^3} + c$

$y_{GS} = c\mathrm{e}^{x^3} - 2$

$y(0) = 1 = c - 2$

$c = 3$

$y_{PS} = 3\mathrm{e}^{x^3} - 2$

4 **a**

$\mu = \mathrm{e}^{-\int\frac{2x}{x^2+1}\mathrm{d}x} = \mathrm{e}^{-\ln(x^2+1)} = \mathrm{e}^{\ln(x^2+1)^{-1}} = \frac{1}{x^2+1}$

b

Multiplying both sides by the integrating factor:

$$(1+x^2)^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy(1+x^2)^{-2} = (1+x^2)^{-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{1+x^2}\right) = \frac{1}{1+x^2}$$

$$\frac{y}{1+x^2} = \arctan x + c$$

$$y_{GS} = (1+x^2)(\arctan x + c)$$

5

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - y^2 = f(x,y)$$

$$y_{n+1} = y_n + h \times f(x_n, y_n)$$

$$y(0) = 2, h = 0.1$$

From GDC: $y(2) \approx 2.45$

6

Maclaurin expansion of $\sin x$:

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)$$

$$\text{Then}(1-x^2)\sin x = (1-x^2)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)\right)$$

$$= x + x^3\left(-\frac{1}{6} - 1\right) + x^5\left(\frac{1}{120} + \frac{1}{6}\right) + -\frac{1}{120}(x^7)$$

$$= x - \frac{7}{6}x^3 + \frac{7}{40}x^5$$

7 **a**

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$$

b

$$\sqrt[3]{\mathrm{e}} = \mathrm{e}^{\frac{1}{3}}$$

$$\approx 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 = \frac{25}{18}$$

8 **a**

$$\arctan u = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 + O(u^7)$$

Substituting $u = 2x$:

$$\arctan 2x = 2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 + O(x^7)$$

b

$$\frac{2x - \arctan 2x}{x^3} = \frac{\frac{8}{3}x^3 - \frac{32}{5}x^5 + O(x^7)}{x^3}$$

$$= \frac{8}{3} + O(x^2)$$

Hence $\lim_{x\to 0}\frac{2x - \arctan 2x}{x^3} = \frac{8}{3}$

9 **a**

$e^u = 1 + u + \frac{1}{2}u^2 + O(u^3)$

Substituting $u = x^2$:

$e^{x^2} = 1 + x^2 + O(x^4)$

b

Taking the indefinite Integral twice:

$\int e^{x^2}\ dx = c + x + \frac{1}{3}x^3 + O(x^5)$

$\int\left(\int e^{x^2}\ dx\right)\ dx = d + cx + \frac{1}{2}x^2 + \frac{1}{12}x^4 + O(x^6)$

If $y'' = e^{x^2}$ then

$y = d + cx + \frac{1}{2}x^2 + \frac{1}{12}x^4 + O(x^6)$

10 **a**

$\frac{dy}{dx} = \cos^2 x - y\tan x\ = f(x, y)$

$y_{n+1} = y_n + h \times f(x_n, y_n)$

$x_0 = 0, y_0 = 2, h = 0.1$

From GDC: $y(0.3) \approx 2.23$

bi

$\mu = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x$

bii

Multiplying both sides by the integrating factor:

$\sec x \frac{dy}{dx} + y \sec x \tan x = \cos x$

$\frac{d}{dx}(y \sec x) = \cos x$

$y \sec x = \sin x + c$

$y_{GS} = (\sin x + c)\cos x$

$y(0) = 2 = c$

$y_{PS} = (2 + \sin x)\cos x$

11 **a**

$\int R^{-1}\ dR = \int -k\ dt$

$\ln|R| = -kt + c$

$R = Ae^{-kt}$

$R(0) = R_0 = A$

$R = R_0e^{-kt}$

b

When $R = \frac{1}{2}R_0, e^{-kt} = \frac{1}{2}$

$t = \frac{1}{k}\ln 2$

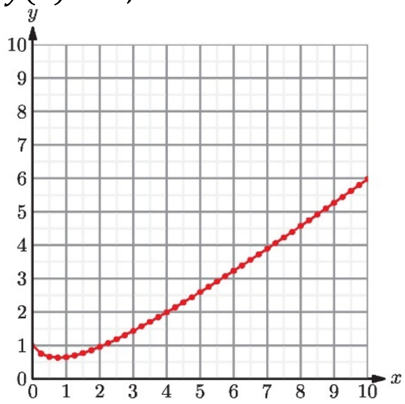
c

The time taken to halve the amount of the substance is independent of the amount at the start; that is, half-life is independent of the amount of the substance involved.

12 **a** Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$y(0) = 1, h = 0.25$



b The minimum value of y is approximately 0.6

13 **a** Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$y(0) = -1, h = 0.1$

From GDC: $y(2) \approx -0.599$

b

$$\int \mathrm{e}^{-y}\ \mathrm{d}y = \int \mathrm{e}^{-x}\ \mathrm{d}x$$

$-\mathrm{e}^{-y} = -\mathrm{e}^{-x} + c$

$y = -\ln(\mathrm{e}^{-x} - c)$

$y(0) = -1 = -\ln(1 - c)$ so $c = 1 - \mathrm{e}$

$y = -\ln(\mathrm{e}^{-x} + \mathrm{e} - 1)$

Then $y(2) = -0.617$

Error $= 0.0178$

c Decreasing the step length is likely to decrease the error

14 **a** Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$y(0) = -1, h = 0.05$

From GDC: $y(1) \approx -0.0392$

b

$\mu = \mathrm{e}^{\int -2x\,\mathrm{d}x} = \mathrm{e}^{-x^2}$

Multiplying both sides by the integrating factor:

$$\mathrm{e}^{-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - 2xe^{-2x^2}y = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-x^2}\right) = 1$$

$y\mathrm{e}^{-x^2} = x + c$

$y_{GS} = (x + c)\mathrm{e}^{x^2}$

$y(0) = -1 = c$

$y_{PS} = (x - 1)\mathrm{e}^{x^2}$

Then $y(1) = 0$

Error $= 0.0392$

15

$\frac{dy}{dx} + y\sec^2 x = \sec^2 x$

$\mu = e^{\int \sec^2 x \, dx} = e^{\tan x}$

Multiplying both sides by the integrating factor:

$e^{\tan x}\frac{dy}{dx} + ye^{\tan x}\sec^2 x = e^{\tan x}\sec^2 x$

$\frac{d}{dx}(ye^{\tan x}) = e^{\tan x}\sec^2 x$

$ye^{\tan x} = e^{\tan x} + c$

$y_{GS} = 1 + ce^{-\tan x}$

$y(0) = 3 = 1 + c$

$c = 2$

$y_{PS} = 1 + 2e^{-\tan x}$

16

Separating variables:

$\int \frac{1}{y^2+1}\, dy = \int \frac{2}{x^2+1}\, dx$

$\arctan y = 2\arctan x + c$

$y = 0$ when $x = 0$

$0 = 0 + c \Rightarrow c = 0$

$\arctan y = 2\arctan x$

$y = \tan(2\, arctan\, x)$

$= \frac{2\tan(\arctan x)}{1-\tan^2(\arctan x)}$ using double angle identity for tan

$= \frac{2x}{1-x^2}$ using $\tan(\arctan x) = x$

17

$\frac{dy}{dx} - 4xy = e^{2x^2}$

$\mu = e^{\int -4x\, dx} = e^{-2x^2}$

Multiplying both sides by the integrating factor:

$e^{-2x^2}\frac{dy}{dx} - 4xye^{-2x^2} = 1$

$\frac{d}{dx}\left(ye^{-2x^2}\right) = 1$

$ye^{-2x^2} = x + c$

$y_{GS} = (x + c)e^{2x^2}$

$y(0) = 4 = c$

$y_{PS} = (x + 4)e^{2x^2}$

18 **a**

Euler's method:

$y(x + h) = y(x) + h \times \frac{dy}{dx}(x)$

$y(1) = 2, h = 0.1$

From GDC: $y(1.3) \approx 3.92$

b

$$\int y^{-1}\ \mathrm{d}y = \int 3x^{-2}\ \mathrm{d}x$$
$$\ln|y| = -3x^{-1} + c$$
$$y = A\mathrm{e}^{-3x^{-1}}$$
$$y(1) = 2 = A\mathrm{e}^{-3} \Rightarrow A = 2\mathrm{e}^{3}$$
$$y = 2\mathrm{e}^{3(1-x^{-1})}$$

c

So $y(1.3) = 3.997$

$$\text{Percentage error} = \frac{|\text{estimated value} - \text{actual value}|}{|\text{actual value}|} \times 100\% = 1.90\%$$

19 **a**

$$(x-1)y' = \cos^2 y$$
$$\int \sec^2 y\ \mathrm{d}y = \int \frac{1}{x-1}\ \mathrm{d}x$$
$$\tan y = \ln|x-1| + c$$
$$y_{GS} = \arctan(\ln|x-1| + c)$$
$$y(0) = 0 = \arctan(c) \Rightarrow c = 0$$
$$y_{PS} = \arctan(\ln|x-1|)$$

b

Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}\big(x, y(x)\big)$$
$$y(0) = 0, h = 0.1, \frac{\mathrm{d}y}{\mathrm{d}x}\big(x, y(x)\big) = \frac{\cos^2 y}{x-1}$$

From GDC: $y(0.5) \approx -0.592$

From part **a**, $y(0.5) = -0.606$

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{|\text{true value}|} \times 100\%$$
$$= 2.39\%$$

20

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.2N\left(1 + 2\sin\left(\frac{\pi t}{6}\right)\right)$$
$$\int \frac{5}{N}\ \mathrm{d}N = \int 1 + 2\sin\left(\frac{\pi t}{6}\right)\ \mathrm{d}t$$
$$5\ln|N| = t - \frac{12}{\pi}\cos\left(\frac{\pi t}{6}\right) + c$$
$$N_{GS} = A\mathrm{e}^{0.2\left(t - \frac{12}{\pi}\cos\left(\frac{\pi t}{6}\right)\right)}$$
$$N(0) = 2 = A\mathrm{e}^{-\frac{2.4}{\pi}} \Rightarrow A = 2\mathrm{e}^{\frac{2.4}{\pi}}$$
$$N_{PS} = 2\mathrm{e}^{0.2\left(t + \frac{12}{\pi}\left(1 - \cos\left(\frac{\pi t}{6}\right)\right)\right)}$$

21 **a**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(1 + x^2 - y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2x + 2x^3$$

b

$\mu = e^{\int 2x \, dx} = e^{x^2}$

Multiplying both sides by the integrating factor:

$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 2xe^{x^2} + 2x^3e^{x^2}$

$\frac{d}{dx}\left(ye^{x^2}\right) = 2xe^{x^2} + 2x^3e^{x^2}$

$ye^{x^2} = e^{x^2} + \int 2x^3e^{x^2} \, dx$

Let $u = x^2, v' = 2xe^{x^2}$ so $u' = 2x, v = e^{x^2}$

Integration by parts: $\int uv' \, dx = uv - \int u'v \, dx$

$\int 2x^3e^{x^2} \, dx = x^2e^{x^2} - \int 2xe^{x^2} \, dx$

$= x^2e^{x^2} - e^{x^2} + c$

$ye^{x^2} = x^2e^{x^2} + c$

$y_{GS} = x^2 + xe^{-x^2}$

22 **a**

$\frac{dy}{dx} = 3\left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2$

This is a homogeneous equation because the derivative can be expressed as a function of the ratio of the variables.

b

Let $y = xw$ so $y' = xw' + w$

$xw' + w = 3w + 2w^2$

$w' = \frac{2w + 2w^2}{x}$

$\int \frac{1}{w(1+w)} \, dw = \int \frac{2}{x} \, dx$

$\int \frac{1}{w} - \frac{1}{1+w} \, dw = \int \frac{2}{x} \, dx$

$\ln\left|\frac{w}{1+w}\right| = 2\ln|x| + c$

$\frac{w}{1+w} = kx^2$

$w(1 - kx^2) = kx^2$

$w = \frac{kx^2}{1 - kx^2} = \frac{x^2}{A - x^2}$

Substituting back with $y = xw$

$y_{GS} = \frac{x^3}{A - x^2}$

$y(2) = 4 = \frac{8}{A - 4} \Rightarrow A = 6$

$y_{PS} = \frac{x^3}{6 - x^2}$

23

Let $f(x) = \mathrm{e}^{2x}$ and $\mathrm{g}(x) = (1-2x)^{\frac{1}{3}}$

Maclaurin expansion for $\mathrm{f}(x)$ is

$$\mathrm{f}(x) = 1 + 2x + \frac{(2x)^2}{2} + O(x^3) = 1 + 2x + 2x^2 + O(x^3)$$

Maclaurin expansion for $\mathrm{g}(x)$ is

$$\mathrm{g}(x) = 1 + \frac{1}{3}(-2x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(-2x)^2 + O(x^3) = 1 - \frac{2}{3}x - \frac{4}{9}x^2 + O(x^3)$$

Then the Maclaurin expansion for $\mathrm{f}(x)\mathrm{g}(x)$ is

$$\mathrm{f}(x)\mathrm{g}(x) = \left(1 + 2x + 2x^2 + O(x^3)\right)\left(1 - \frac{2}{3}x - \frac{4}{9}x^2 + O(x^3)\right)$$
$$= 1 + \left(2 - \frac{2}{3}\right)x + \left(2 - \frac{4}{3} - \frac{4}{9}\right)x^2 + O(x^3)$$

The term in x^2 is $\frac{2}{9}x^2$

24

Let $\mathrm{f}(x) = (1-x)^{-1}$

Maclaurin expansion for $\mathrm{f}(x)$ is

$\mathrm{f}(x) = 1 + x + x^2 + x^3 + O(x^4)$

Let $\mathrm{g}(x) = \ln\left(f(x)\right)$

Maclaurin expansion for $\mathrm{g}(x)$ is

$$\mathrm{g}(x) = \left(x + x^2 + x^3 + O(x^4)\right) - \frac{1}{2}\left(x + x^2 + x^3 + O(x^4)\right)^2$$
$$+ \frac{1}{3}\left(x + x^2 + x^3 + O(x^4)\right)^3 + O(x^4)$$
$$= \left(x + x^2 + x^3 + O(x^4)\right) - \frac{1}{2}\left(x^2 + 2x^3 + O(x^4)\right) + \frac{1}{3}\left(x^3 + O(x^4)\right) + O(x^4)$$
$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

25

Let $f(x) = \cos x$ and $\mathrm{g}(x) = (1-x^2)^{-\frac{1}{2}}$

Maclaurin expansion for $\mathrm{f}(x)$ is

$$\mathrm{f}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$$

Maclaurin expansion for $\mathrm{g}(x)$ is

$$\mathrm{g}(x) = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + O(x^6) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6)$$

Then the Maclaurin expansion for $\mathrm{f}(x)\mathrm{g}(x)$ is

$$\mathrm{f}(x)\mathrm{g}(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\right)\left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6)\right)$$
$$= 1 + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{3}{8} - \frac{1}{4} + \frac{1}{24}\right)x^4 + O(x^6)$$

The term in x^4 is $\frac{4}{24}x^4 = \frac{1}{6}x^4$

26 **a**

$y = \arcsin x$

$y' = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

$y'' = x(1-x^2)^{-\frac{3}{2}}$

$y''' = (1-x^2)^{-\frac{3}{2}} + (3x^2)(1-x^2)^{-\frac{5}{2}} = (1+2x^2)(1-x^2)^{-\frac{5}{2}}$

b

$y(0) = 0$

$y'(0) = 1$

$y''(0) = 0$

$y'''(0) = 1$

Maclaurin series:

$y = y(0) + xy'(0) + \dfrac{x^2}{2!}y''(0) + \dfrac{x^3}{3!}y'''(0) + O(x^4)$

$y = x + \dfrac{1}{6}x^3 + O(x^4)$

c

$\sin x = x - \dfrac{1}{6}x^3 + O(x^5)$

$\lim_{x\to 0}\left\{\dfrac{\arcsin x - \sin x}{x^3}\right\} = \lim_{x\to 0}\left\{\dfrac{\frac{1}{3}x^3 + O(x^5)}{x^3}\right\} = \dfrac{1}{3}$

27

$\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + x^2y^2 = x$

So $\dfrac{d^2y}{dx^2} = x - x\dfrac{dy}{dx} - x^2y^2$

Then $\dfrac{d^3y}{dx^3} = 1 - x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2xy^2 - 2x^2y\dfrac{dy}{dx}$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$

$y(0) = a_0 = 2$

$y'(0) = a_1 = 4$

$y''(0) = 2a_2 = 0 - 0(4) - 0(2^2) = 0$

$y'''(0) = 6a_3 = 1 - 0 - 4 - 2(0)(4) + 2(0)(2)(4) = -3$

$y = 2 + 4x - \dfrac{1}{2}x^3 + O(x^4)$

28

$\dfrac{dy}{dx} = x + 2 - xe^y$

So $\dfrac{d^2y}{dx^2} = 1 - e^y\left(x\dfrac{dy}{dx} + 1\right)$

Let $y = a_0 + a_1x + a_2x^2 + O(x^3)$

$y(0) = a_0 = 1$

$y'(0) = a_1 = 0 + 2 - 0(e^1) = 2$

$y''(0) = 2a_2 = 1 - e^1(0(2) + 1) = 1 - e$

$y = 1 + 2x + \dfrac{1-e}{2}x^2 + O(x^3)$

29 **a**

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$$

$\ln|y| = \ln|x| + k = \ln|cx|$

$y = cx$

b

Let $y = vx$ so $y' = v + xv'$

$v + xv' = v$

$xv' = 0$

$v' = 0$

$v = c$

Substituting back:

$\frac{y}{x} = c$

$y = cx$

c

$y' - x^{-1}y = 0$

$\mu = \mathrm{e}^{\int -x^{-1}\,\mathrm{d}x} = \mathrm{e}^{-\ln x} = x^{-1}$

Multiplying both sides by the integrating factor:

$x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} - x^{-2}y = 0$

$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}y) = 0$

$x^{-1}y = c$

$y = cx$

d)

$y(2) = 20 = 2c \Rightarrow c = 10$

$y(5) = 50$

30 **ai**

$yy' = \cos 2x$

If $y = \cos x + \sin x$ then $y' = \cos x - \sin x$

$yy' = (\cos x + \sin x)(\cos x - \sin x) = \cos^2 x - \sin^2 x = \cos 2x$

(using the double angle formula for cosine)

Hence $y = \cos x + \sin x$ satisfies the differential equation.

aii Separating variables:

$$\int y\ \mathrm{d}y = \int \cos 2x\ \mathrm{d}x$$

$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + c$

$y_{GS} = \pm\sqrt{\sin 2x + c}$

aiii

$(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + 2\sin x\cos x = 1 + \sin 2x$

Therefore for $c = 1$, the answer to **ii** gives the result seen in part **i**, with the positive square root.

bi

$y\left(\frac{\pi}{4}\right) = 2 = \sqrt{\sin\left(\frac{\pi}{2}\right) + c} = \sqrt{1 + c} \Rightarrow c = 3$

$y_{PS} = \sqrt{\sin 2x + 3}$

Since $\sin 2x$ has range $[-1,1]$, the range of $\mathrm{g}(x)$ is $[\sqrt{2}, 2]$

bii

$$\text{Area} = \int_0^{\frac{\pi}{2}} \mathrm{g}(x)\ \mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin 2x + 3}\ \mathrm{d}x$$
$$= 2.99\ (\text{GDC})$$

biii

The volume required equals the volume of revolution of the curve about the x-axis less the volume of a disc of radius 1 and axis length $\frac{\pi}{2}$.

$$V = \pi \int_0^{\frac{\pi}{2}} y^2\ \mathrm{d}x - \pi(1)^2\left(\frac{\pi}{2}\right)$$
$$= \pi \int_0^{\frac{\pi}{2}} \sin 2x + 3\ \mathrm{d}x - \frac{\pi^2}{2}$$
$$= \pi\left[3x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} - \frac{\pi^2}{2}$$
$$= \pi\left(\frac{3\pi}{2} + 2\right) - \frac{\pi^2}{2}$$
$$= \pi^2 + 2\pi$$
$$\approx 13.0$$

31

Separating variables:

$$\int \frac{1}{1+y^2}\ \mathrm{d}y = \int \frac{1}{1+x^2}\ \mathrm{d}x$$
$$\arctan y = \arctan x + c$$

Using the compound angle formula for the tangent function:

$$y = \tan(\arctan x + c) = \frac{x + \tan c}{1 - x\tan c}$$

Since $\tan c$ can take any value for unknown c, we can simplify to

$$y_{GS} = \frac{x + k}{1 - kx}$$
$$y(0) = 1 = k$$
$$y = \frac{x + 1}{1 - x}$$

32 **a**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y}{x}\right) + \frac{1}{\ln y - \ln x} = \left(\frac{y}{x}\right) + \frac{1}{\ln\left(\frac{y}{x}\right)}$$

This is a homogeneous equation because the derivative can be expressed as a function of the ratio of the variables.

b

Let $y = xw$ so that $y' = xw' + w$

$$xw' + w = w + \frac{1}{\ln w}$$

$$xw' = \frac{1}{\ln w}$$

Separating variables:

$$\int \ln w \ \mathrm{d}w = \int \frac{1}{x} \ \mathrm{d}x$$

$$w \ln w - w = \ln|x| + c = \ln|kx|$$

Substituting back:

$$\frac{y}{x}\left(\ln\left(\frac{y}{x}\right) - 1\right) = \ln|kx|$$

$y(1) = 1$ implies $k = e^{-1}$

$$y\left(\ln\left(\frac{y}{x}\right) - 1\right) = x\ln\left(\frac{x}{e}\right)$$

33 **a**

$\frac{1+v}{9-v^2} = \frac{1+v}{(3+v)(3-v)} = \frac{A}{3+v} + \frac{B}{3-v}$ for some constants A and B

Multiplying through by the denominator of the LHS:

$$1 + v = A(3 - v) + B(3 + v)$$

$$v = 3: 4 = 6B \Rightarrow B = \frac{2}{3}$$

$$u = -3: \ -2 = 6A \Rightarrow A = -\frac{1}{3}$$

$$\frac{1+v}{9-v^2} = \frac{1}{3}\left(\frac{2}{3-v} - \frac{1}{3+v}\right)$$

b

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x + y}{x + y} = \frac{9 + \left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)}$$

This is a homogeneous equation because the derivative can be expressed as a function of the ratio of the variables.

c

Let $y = xw$ so that $y' = xw' + w$

$$xw' + w = \frac{9 + w}{1 + w}$$

$$xw' = \frac{9 - w^2}{1 + w}$$

$$\int \frac{1 + w}{9 - w^2} \ \mathrm{d}w = \int \frac{1}{x} \ \mathrm{d}x$$

Using part **a** to split the left integrand into partial fractions:

$$\frac{1}{3}\int \left(\frac{2}{3-w} - \frac{1}{3+w}\right) \mathrm{d}w = \ln|x| + c$$

$$-2\ln|3 - w| - \ln|3 + w| = 3\ln|x| + c = \ln|Ax^3|$$

$$\ln|Ax^3(3 - w)^2(3 + w)| = 0$$

$$Ax^3(3 - w)^2(3 + w) = 1$$

Substituting back to y:

$Ax^3\left(3-\frac{y}{x}\right)^2\left(3+\frac{y}{x}\right)=1$

$(3x-y)^2(3x+y)=A^{-1}$

$(3x-y)(3x-y)(3x+y)=A^{-1}$

$(3x-y)(9x^2-y^2)=A^{-1}$

Therefore, $(3x-y)(9x^2-y^2)$ is a constant in any solution.

34

Let $z=2x-3y$ so $z'=2-3y'$

Then $y'=\frac{(2-z')}{3}$

Substituting:

$\frac{(z+3)(2-z')}{3}=z+1$

$2(z+3)-(z+3)z'=3z+3$

$(z+3)z'=3-z$

$\int\frac{z+3}{3-z}\,\mathrm{d}z=\int 1\ \mathrm{d}x$

$\int\frac{6}{3-z}-1\ \mathrm{d}z=\int 1\ \mathrm{d}x$

$-6\ln|3-z|-z=x+c$

When $x=1, y=1$ and $z=-1$

$-6\ln 4+1=1+c\Rightarrow c=-6\ln 4$

$-6\ln\left|\frac{z-3}{4}\right|-z=x$

$3y-2x-6\ln\left|\frac{2x-3y-3}{4}\right|=x$

$6\ln\left|\frac{2x-3y-3}{4}\right|=3(y-x)$

$\left(\frac{2x-3y-3}{4}\right)^2=\mathrm{e}^{y-x}$

$16\mathrm{e}^{y-x}=(2x-3y-3)^2$

35

$y'+\frac{y}{x}=\sqrt{\frac{y}{x}}$

Let $y=u^2$ so $y'=2u\,u'$

$2u\,u'+\frac{u^2}{x}=\frac{u}{\sqrt{x}}$

$u+\frac{1}{2x}u=\frac{1}{2\sqrt{x}}$

Integrating factor $\mu(x)$

$\mu=\mathrm{e}^{\int 0.5x^{-1}\,\mathrm{d}x}=\mathrm{e}^{0.5\ln x}=\sqrt{x}$

Multiplying both sides by the integrating factor:

$\sqrt{x}\,u'+\frac{1}{2\sqrt{x}}u=\frac{1}{2}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(u\sqrt{x}\right)=\frac{1}{2}$

$u\sqrt{x}=\frac{1}{2}x+c$

$$u = \frac{\sqrt{x}}{2} + \frac{c}{\sqrt{x}} = \frac{x + 2c}{2\sqrt{x}}$$

$$y_{GS} = u^2 = \frac{(x + 2c)^2}{4x}$$

$$y(1) = 0 = \frac{(1 + 2c)^2}{4} \Rightarrow 2c = -1$$

$$y_{PS} = \frac{(x - 1)^2}{4x}$$

36

Let $\mathrm{f}(x) = \ln(6 + 5x) = \ln 6 + \ln\left(1 + \frac{5}{6}x\right)$

Let $\mathrm{g}(x) = \ln(6 - 5x) = \ln 6 + \ln\left(1 - \frac{5}{6}x\right)$

Maclaurin expansion for $\mathrm{f}(x)$ is

$$\mathrm{f}(x) = \ln 6 + \frac{5}{6}x - \frac{1}{2}\left(\frac{5}{6}x\right)^2 + O(x^3) = \ln 6 + \frac{5}{6}x - \frac{25}{72}x^2 + O(x^3)$$

Maclaurin expansion for $\mathrm{g}(x)$ is

$$\mathrm{g}(x) = \ln 6 + \left(-\frac{5}{6}x\right) - \frac{1}{2}\left(-\frac{5}{6}x\right)^2 + O(x^3) = \ln 6 - \frac{5}{6}x - \frac{25}{72}x^2 + O(x^3)$$

$$\ln\left(\frac{6 + 5x}{6 - 5x}\right) = \mathrm{f}(x) - \mathrm{g}(x) = \frac{5}{3}x + O(x^3) \approx \frac{5}{3}x \text{ for small values of } x$$

37 **a**

Let $\mathrm{f}(x) = \ln(\sin x + \cos x)$

$$\mathrm{f}'(x) = \frac{\cos x - \sin x}{\sin x + \cos x} = \frac{1 - \tan x}{\tan x + 1} = \frac{2}{\tan x + 1} - 1$$

$$\mathrm{f}''(x) = -\frac{2\sec^2 x}{(\tan x + 1)^2} = -\frac{2}{(\sin x + \cos x)^2}$$

$$\mathrm{f}'''(x) = \frac{4(\cos x - \sin x)}{(\sin x + \cos x)^3}$$

b

$f(0) = 0$

$f'(0) = 1$

$f''(0) = -2$

$f'''(0) = 4$

Maclaurin series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + O(x^4)$$

$$\mathrm{f}(x) = x - x^2 + \frac{2}{3}x^3 + O(x^4)$$

Let $\mathrm{g}(x) = \arctan 2x$

$$\mathrm{g}(x) = (2x) - \frac{1}{3}(2x)^3 + O(x^5) = 2x - \frac{8}{3}x^3 + O(x^5)$$

$$\begin{aligned}\lim_{x\to 0}\left\{\frac{\ln(\sin x + \cos x)}{\arctan 2x}\right\} &= \lim_{x\to 0}\left\{\frac{x - x^2 + \frac{2}{3}x^3 + O(x^4)}{2x - \frac{8}{3}x^3 + O(x^5)}\right\} \\ &= \lim_{x\to 0}\left\{\frac{1 + O(x)}{2 + O(x^2)}\right\} \\ &= \frac{1}{2}\end{aligned}$$

38 **a**

$\mathrm{f}(x) = \mathrm{e}^{\sin x}$

$\mathrm{f}'(x) = \cos x \ \mathrm{e}^{\sin x}$

Using product rule: Let $u = \cos x$ and $v = \mathrm{e}^{\sin x}$ so $u' = -\sin x$ and $v' = \cos x \ \mathrm{e}^{\sin x}$

$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = uv' + u'v = \cos^2 x \, \mathrm{e}^{\sin x} - \sin x \, \mathrm{e}^{\sin x}$

$\mathrm{f}''(x) = \mathrm{e}^{\sin x}(\cos^2 x - \sin x)$

b

$\mathrm{f}(0) = 1$

$\mathrm{f}'(0) = 1$

$\mathrm{f}''(0) = 1$

$\mathrm{f}'''(0) = 0, \mathrm{f}^{(4)}(0) = -3$

Maclaurin series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \frac{x^4}{4!}f^{(4)}(0) + O(x^5)$$

$$\mathrm{f}(x) = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^5)$$

c

The Maclaurin series for $\mathrm{g}(x) = \mathrm{e}^x$ is $\mathrm{g}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)$

$$\lim_{x\to 0}\left\{\frac{\mathrm{e}^x - \mathrm{e}^{\sin x}}{x^3}\right\} = \lim_{x\to 0}\left\{\frac{\frac{1}{6}x^3 + O(x^4)}{x^3}\right\} = \lim_{x\to 0}\left\{\frac{1}{6} + O(x)\right\} = \frac{1}{6}$$

39 **a**

$$\lim_{t\to 0}\frac{\sin t}{t} = 1$$

Then $\lim_{x\to 0}\frac{\sin(4x^2)}{4x^2} = 1$

And $\lim_{x\to 0}\frac{\sin(9x^2)}{9x^2} = 1$ so $\lim_{x\to 0}\frac{\sin(9x^2)}{4x^2} = \frac{9}{4}$

Summing these two limits:

$$\lim_{x\to 0}\frac{\sin(4x^2) - \sin(9x^2)}{4x^2} = 1 - \frac{9}{4} = -\frac{5}{4}$$

b

Maclaurin series for $\mathrm{f}(x) = \sin x$ is

$$\mathrm{f}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Then since $\mathrm{g}(x) = \mathrm{f}(x^2)$, the series for $\mathrm{g}(x)$ is

$$\mathrm{g}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

c

Let $I = \int_0^1 g(x)\ dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{4n+2}\ dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{x^{4n+3}}{4n+3}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3}$$

$$= \sum_{n=0}^{\infty} (-1)^n a_n$$

for a sequence $\{a_n\}$ where

$$a_n = \frac{1}{(4n+3)(2n+1)!}$$

Since the sequence is decreasing and the series is alternating, the value to which the series converges is approximately S_N with an absolute error less than a_{N+1} for each N.
If the sum approximation is to be accurate to $4DP$, require the absolute error to be less than 0.00005

$$a_{N+1} = \frac{1}{(4N+7)(2N+3)!} < 0.00005$$

$$a_{1+1} = \frac{1}{11 \times 5!} = 0.000076$$

$$a_{2+1} = \frac{1}{15 \times 7!} = 0.000013 < 0.00005$$

So $N = 2$, and therefore the series is accurate to $4DP$ after three terms: $a_0 - a_1 + a_2$

40

$y'' = 2xy' - y$

$y''' = 2y' - y' + 2xy'' = 2xy'' + y'$

$y^{(4)} = 2y'' + 2xy''' + y'' = 2xy''' + 3y''$

$y^{(5)} = 2y''' + 2xy^{(4)} + 3y''' = 2xy^{(4)} + 5y'''$

It appears that the sequence of higher derivatives follows $y^{(n+2)} = 2xy^{(n+1)} + (2n-1)y^{(n-1)}$

Proving this by induction:

Proposition: $y^{(n+2)} = 2xy^{(n+1)} + (2n-1)y^{(n)}$ for $n \geq 0$

Base case $n = 0$: $y^{(2)} = 2xy' - y = 2xy^{(1)} + (2(0)-1)y^{(0)}$ so the proposition is true for $n = 0$

Inductive step: Assume the proposition is true for integer $n = k \geq 0$

So $y^{(k+2)} = 2xy^{(k+1)} + (2k-1)y^{(k)}$

Working towards: $y^{(k+3)} = 2xy^{(k+2)} + (2k+1)y^{(k+1)}$

$$\begin{aligned} y^{(k+3)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{(k+2)}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2xy^{(k+1)} + (2k-1)y^{(k)}\right) \quad \text{(by assumption)} \\ &= 2xy^{(k+2)} + 2y^{(k+1)} + (2k-1)y^{(k+1)} \\ &= 2xy^{(k+2)} + (2k+1)y^{(k+1)} \end{aligned}$$

So the proposition is true for $n = k + 1$

Conclusion:

The proposition is true for $n = 0$, and, if true for $n = k$, it is also true for $n = k + 1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y(0) = a_0 = 0$

$y'(0) = a_1 = 1$

$y''(0) = 2a_2 = 0$

$y^{(k+2)}(0) = (k+2)!\, a_{k+2} = (2k-1)y^{(k)}$ from the induction above

$$a_{k+2} = \frac{2k-1}{(k+2)!} y^{(k)}(0)$$

Even coefficients:

$a_{2n} = 0$ for all integer n

Odd coefficients: substituting $k = 2n - 1$ into the formula above for $n \geq 1$

$$\begin{aligned} a_{2n+1} &= \frac{1}{(2n+1)!}(4n-3)y^{(2n-1)}(0) \\ &= \frac{1}{(2n+1)!}(4n-3)(2n-1)!\, a_{2n-1} \\ &= \frac{4n-3}{(2n+1)(2n)} a_{2n-1} \end{aligned}$$

Iterating this back to $a_1 = 1$:

$$a_{2n+1} = \frac{1 \times 5 \times 9 \times \ldots \times (4n-3)}{(2n+1)!}$$

$$y = x + \sum_{n=1} \frac{1 \times 5 \times 9 \ldots (4n-3)}{(2n+1)!} x^{2n+1} = \sum_{n=0} \frac{1 \times 5 \times 9 \ldots (4n+1)}{(2n+1)!} x^{2n+1}$$